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The large sample distribution of the Shapiro–Wilk statistic and its variants under Type I or Type II censoring

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Abstract: The original Shapiro–Wilk statistic is extended for testing normality when the observations are Type I or Type II censored. We determine its large sample limit distribution under Type I or Type II censoring. This censored data limit distribution has an interesting relation to the complete sample solution and is obtained from it by replacing each Hermite polynomial with a censored data form. The same limit distribution also applies to several variants of the Shapiro–Wilk statistic which are related to the correlation coefficient associated with a normal probability plot.

Keywords: Asymptotic distributions, Type I and Type II censoring, correlation coefficient tests of normality, modified Shapiro–Wilk statistics, normal probability plot

1. Introduction

Let $X = [X_{1n}, \dots, X_{nn}]'$ be the vector of order statistics from a random sample of size n. A commonly applied check for normality is to calculate the Shapiro–Wilk (1965) statistic, W,

$$W = (\boldsymbol{m}'V^{-1}X)^2 \Big/ \left[\boldsymbol{m}'V^{-1}V^{-1}\boldsymbol{m} \sum_{i=1}^{n} (X_{in} - \bar{X})^2\right] \tag{1.1}$$

where $\boldsymbol{m}$, V are the expectation vector and covariance matrix of the order statistics from a sample of n standard normal random variables. One popular variant, the Shapiro and Francia (1972) statistic, is obtained by replacing V by the identity $\boldsymbol{I}$.

Filliben (1975) and Ryan and Joiner (1973) noted that the Shapiro–Francia statistic could be written as the square of the correlation coefficient associated with a normal probability plot. Filliben proposed calculating the correlation coefficient from a plot of X_{in} versus the median, M_{in}, of the ith order statistic from a sample of n standard normal variables and Ryan and Joiner (1973) proposed using the closely related score $H_{in} = H(i/(n+1))$ where $H(\ )$ is the inverse of the standard normal c.d.f.

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An extension of these latter statistics to Type II censored data takes the form

$$r(\boldsymbol{X}, \boldsymbol{b}, \delta)=\frac{\sum_{i=1}^{[n\delta]}\left(X_{in}-\bar{X}_{n,\delta}\right)\left(b_{in}-\bar{b}_{n,\delta}\right)}{\left(\sum_{i=1}^{[n\delta]}\left(X_{in}-\bar{X}_{n,\delta}\right)^{2} \sum_{i=1}^{[n\delta]}\left(b_{in}-\bar{b}_{n,\delta}\right)^{2}\right)^{1/2}} \tag{1.2}$$

where $0<\delta<1$, [ ] is the greatest integer function,

$$\bar{b}_{n,\delta}=\sum_{i=1}^{[n\delta]} b_{in}/[n\delta], \qquad \bar{X}_{n,\delta}=\sum_{i=1}^{[n\delta]} X_{in}/[n\delta]$$

and $b_{in}=H_{in}$, M_{in}, or a related score.

The Type I censored data version has the same form except that the upper limit of summation and divisor for the mean is replaced by the random number of order statistics observed.

Our censored data extension of the Shapiro–Wilk statistic is obtained by replacing $\boldsymbol{b}$ in (1.2) by $\boldsymbol{a} \equiv \boldsymbol{V}^{-1}\boldsymbol{m}$.

Smith and Bain (1976) proposed (1.2) with the normal scores H_{in}.

The major result in this paper, the limit distribution of $r(\boldsymbol{X}, \boldsymbol{H}, \delta)$, is presented in Theorem 1 below. Leslie, Stephens and Fotopoulos (1986) established the asymptotic equivalence of $r(\boldsymbol{X}, \boldsymbol{H}, 1)$ and $r(\boldsymbol{X}, \boldsymbol{a}, 1)$. Verrill and Johnson (1987, 1988a) established the asymptotic equivalence of $r(\boldsymbol{X}, \boldsymbol{H}, \delta)$ with $r(\boldsymbol{X}, \boldsymbol{b}, \delta)$ for a wide class of alternative scores, $\boldsymbol{b}$ (including the Shapiro–Wilk scores, $\boldsymbol{a}$). Verrill and Johnson (1988a) also provide tables of critical values and discuss a small sample power study.

Before proceeding with the statement and proof of Theorem 1, we need to define a number of constants. The definitions are analogous to those in DeWet and Venter (1972, 1973). Let

$$t_{n,\delta}^{2}=\sum_{i=1}^{[n\delta]}\left(H_{in}-\bar{H}_{n,\delta}\right)^{2}/[n\delta]. \tag{1.3}$$

Let

$$H_{in}'=H'\left(\frac{i}{n+1}\right)=\frac{1}{\phi(H_{in})}$$

where $\phi(x)$ is the standard normal p.d.f.

Let

$$\begin{aligned}
a_{n,\delta}^{0} &= \sum_{i=1}^{[n\delta]} \frac{(i/(n+1))(1-i/(n+1))}{\phi^{2}(H_{in})(n+1)}, \\
a_{n,\delta}' &= \sum_{r=1}^{[n\delta]} \sum_{s=1}^{[n\delta]} \frac{T_{rsn}}{\phi(H_{rn})\phi(H_{sn})[n\delta](n+1)}, \\
a_{n,\delta}'' &= \sum_{r=1}^{[n\delta]} \sum_{s=1}^{[n\delta]} \frac{\left(H_{rn}-\bar{H}_{n,\delta}\right)\left(H_{sn}-\bar{H}_{n,\delta}\right)T_{rsn}}{\phi(H_{rn})\phi(H_{sn})[n\delta](n+1)t_{n,\delta}^{2}},
\end{aligned} \tag{1.4}$$

where

$$T_{rsn}=T\left(\frac{r}{n+1}, \frac{s}{n+1}\right)$$

and

$$T(x, y)=\begin{cases} x(1-y) & \text{if } x \leqslant y, \\ (1-x)y & \text{if } x > y. \end{cases}$$

Define

$$a_{n,\delta}=a^0_{n,\delta}-a'_{n,\delta}-a''_{n,\delta}. \tag{1.5}$$

Further, set

$$K_{1,\delta}=\frac{1}{\delta}\int_0^{\delta} H(x)\,\mathrm{d}x, \qquad K_{3,\delta}=\frac{1}{\delta}\int_0^{\delta}(H(x)-K_{1,\delta})^2\,\mathrm{d}x,$$
$$K_{c,\delta}=\int_{\delta}^{1}\frac{(1-x)^2}{\phi^2(H(x))}\,\mathrm{d}x, \qquad K_{d,\delta}=\int_{\delta}^{1}\frac{(1-x)}{\phi(H(x))}\,\mathrm{d}x, \qquad K_{e,\delta}=\int_{\delta}^{1}\frac{(1-x)H(x)}{\phi(H(x))}\,\mathrm{d}x, \tag{1.6}$$

and then let

$$J_{22,\delta}=\frac{1}{2}-\frac{1}{2\delta K_{3,\delta}}, \qquad J_{11,\delta}=1-\frac{1}{\delta}-\frac{K^2_{1,\delta}}{\delta K_{3,\delta}},$$
$$J_{21,\delta}=2^{1/2}\frac{K_{1,\delta}}{\delta K_{3,\delta}}, \qquad J_{20,\delta}=-\frac{2^{1/2}(K_{e,\delta}-K_{1,\delta}K_{d,\delta})}{\delta K_{3,\delta}},$$
$$J_{10,\delta}=\frac{2(K_{1,\delta}(K_{e,\delta}-K_{1,\delta}K_{d,\delta})-K_{d,\delta}K_{3,\delta})}{\delta K_{3,\delta}}, \tag{1.7}$$
$$J_{00,\delta}=-\frac{(K_{e,\delta}-K_{1,\delta}K_{d,\delta})^2+K_{3,\delta}K^2_{d,\delta}}{\delta K_{3,\delta}}-K_{c,\delta}.$$

2. The limit distribution

Our approach extends the methods of DeWet and Venter (1972, 1973) who established the asymptotic theory for the complete sample case.

The limit distribution is expressed in terms of truncated versions of the Hermite polynomials:

$$g_m(x)=\begin{cases} 0 & \text{for } x=0,1, \\ \dfrac{1}{(2^m m!)^{1/2}}h_m\left(\dfrac{H(x)}{2^{1/2}}\right) & \text{for } x\in(0,\delta], \\ g_m(\delta) & \text{for } x\in(\delta,1), \end{cases} \tag{2.1}$$

where h_m, $m=0, 1,\ldots,$ is the mth Hermite polynomial (see Rainville, 1960). Note that we have suppressed the dependence of $g_m(x)$ on δ.

Theorem 1. *Let $t^2_{n,\delta}$ be given by* (1.3), *$a_{n,\delta}$ by* (1.5), *$K_{3,\delta}$ by* (1.6), *and the constants $J_{ij,\delta}$, $i\geqslant j=0, 1, 2$, by* (1.7). *For random samples from a normal distribution,*

$$2[n\delta](1-r(X, H, \delta))-a_{n,\delta}/t^2_{n,\delta}\xrightarrow{\text{D}} Y_{\delta}/K_{3,\delta}$$

where

$$Y_\delta = \sum_{m=3}^{\infty} \frac{1}{m}\left(W_m^2 - E(W_m^2)\right) + \sum_{0 \leqslant j \leqslant i \leqslant 2} J_{ij,\delta}\left(W_iW_j - E(W_iW_j)\right)$$

and for every M, the random variables $W_0, W_1, \ldots, W_M$ have a joint multivariate normal distribution with expectation vector $\mathbf{0}$ *and covariance matrix $\Sigma_{(M+1)\times(M+1)} = (\sigma_{ij})$, where $\sigma_{ij} = \int_0^1 g_i(x)g_j(x)\,\mathrm{d}x$ and the g_m are defined in* (2.1).

Remarks. (1) The same limit distribution applies to the statistic $r(\boldsymbol{X}, \boldsymbol{b}, \delta)$ with, for example, $b_{in} = M_{in}$, m_{in}, a_{in}, or the Weisberg–Bingham (1975) scores. (See Verrill and Johnson (1987, 1988a).)

(2) The case of no censoring corresponds to $\delta = 1$ and the orthogonality of the Hermite polynomials then implies that the covariance terms $\sigma_{ij} = 0$, $i \neq j$. The censoring introduces correlations among the terms of the series for Y_δ. Also, with no censoring, all of the $J_{ij,\delta}$ equal zero so our representation reduces to that given by DeWet and Venter (1972, 1973).

(3) The limit distribution also applies to the squared correlation statistic. Replace $2[n\delta](1 - r(\boldsymbol{X}, \boldsymbol{H}, \delta))$ by $[n\delta](1 - r^2(\boldsymbol{X}, \boldsymbol{H}, \delta))$.

(4) The same limit result is obtained for the Type I censored data correlation statistic. (See Verrill and Johnson (1988a, equation (4.4)) and Verrill (1981) for details on the equivalence of the limits under Type I and Type II censoring.)

Proof of Theorem 1. We present the main steps for extending DeWet and Venter (1972, 1973) to the case of Type II censoring. More details and the Type I censoring solution are given in Verrill (1981).

Because $r(\boldsymbol{X}, \boldsymbol{H}, \delta)$ is invariant under location and scale changes, without loss of generality, we take $X_{1n} < \cdots < X_{nn}$ to be standard normal order statistics. Following DeWet and Venter (1972), we expand $H(\ )$ to obtain

$$X_{in} = H(U_{in}) = H_{in} + H'\left(\frac{i}{n+1}\right)\left(U_{in} - \frac{i}{n+1}\right) + \tfrac{1}{2}H''(U_{in}^*)\left(U_{in} - \frac{i}{n+1}\right)^2$$

where U_{in} is the ith order statistic from a uniform distribution, and U_{in}^* lies between U_{in} and $i/(n+1)$. After some algebra, we obtain

$$\sum_{i=1}^{[n\delta]} \left(X_{in} - \overline{X}_{n,\delta}\right)^2 t_{n,\delta}^2\left(1 - r^2(\boldsymbol{X}, \boldsymbol{H}, \delta)\right)$$
$$= t_{n,\delta}^2\left(Q_{n,\delta} - [n\delta]y_{n,\delta}^2 - [n\delta]W_{n,\delta}^2/t_{n,\delta}^2\right) + R_{n,\delta} \tag{2.2}$$

where

$$Q_{n,\delta} = \sum_{i=1}^{[n\delta]}\left(U_{in} - \frac{i}{n+1}\right)^2\left(H'\left(\frac{i}{n+1}\right)\right)^2,$$

$$[n\delta]y_{n,\delta}^2 = \frac{\left(\sum_{i=1}^{[n\delta]}\left(U_{in} - \frac{i}{n+1}\right)H'\left(\frac{i}{i+1}\right)\right)^2}{[n\delta]},$$

$$[n\delta]W_{n,\delta}^2 = \frac{\left(\sum_{i=1}^{[n\delta]}\left(U_{in} - \frac{i}{n+1}\right)\left(H_{in} - \overline{H}_{n,\delta}\right)H'\left(\frac{i}{n+1}\right)\right)^2}{[n\delta]}.$$

The remainder term, $R_{n,\delta}$, which contains the higher order derivatives, converges to zero in probability.

The next important step is to employ the well known relation

$$U_{in} = \sum_{j=1}^{i} Z_j^* / \sum_{j=1}^{n+1} Z_j^*$$

between uniform order statistics and independent standard exponential variables. Setting $Z_j = Z_j^* - 1$, from (2.2) we have

$$\sum_{i=1}^{[n\delta]} (X_{in} - \overline{X}_{n,\delta})^2 (1 - r^2(\boldsymbol{X}, \boldsymbol{H}, \delta)) - a_{n,\delta}$$
$$= \left(\frac{n+1}{\sum_{j=1}^{n+1} (Z_j + 1)} \right)^2 T_{n,\delta} - a_{n,\delta} + R_{n,\delta}/t_{n,\delta}^2 \tag{2.3}$$

where

$$T_{n,\delta} = \sum_{k=1}^{n+1} \sum_{l=1}^{n+1} (c_{kln,\delta} - d_{kln,\delta} - e_{kln,\delta}) \frac{Z_k}{(n+1)^{1/2}} \frac{Z_l}{(n+1)^{1/2}},$$

the Z_k are independent and identically distributed as standard negative exponentials minus 1,

$$c_{kln,\delta} = \frac{\sum_{i=1}^{[n\delta]} (H'_{in})^2 \psi\left(\frac{k}{n+1}, \frac{i}{n+1}\right) \psi\left(\frac{l}{n+1}, \frac{i}{n+1}\right)}{n+1}, \tag{2.4}$$

$$d_{kln,\delta} = \frac{\sum_{i=1}^{[n\delta]} H'_{in} \psi\left(\frac{k}{n+1}, \frac{i}{n+1}\right)}{n+1} \frac{\sum_{j=1}^{[n\delta]} H'_{jn} \psi\left(\frac{l}{n+1}, \frac{j}{n+1}\right)}{n+1} \frac{n+1}{[n\delta]},$$

$$e_{kln,\delta} = \frac{\sum_{i=1}^{[n\delta]} H'_{in} (H_{in} - \overline{H}_{n,\delta}) \psi\left(\frac{k}{n+1}, \frac{i}{n+1}\right)}{n+1} \cdot \frac{\sum_{j=1}^{[n\delta]} H'_{jn} (H_{jn} - \overline{H}_{n,\delta}) \psi\left(\frac{l}{n+1}, \frac{j}{n+1}\right)}{n+1} \frac{n+1}{[n\delta] t_{n,\delta}^2}$$

and

$$\psi(x, y) = \begin{cases} 1 - y & \text{if } x \leqslant y, \\ -y & \text{if } x > y. \end{cases}$$

A straightforward, but tedious analysis establishes several needed facts (see Verrill, 1981, Chapter 2 for details):

(i) $t_{n,\delta}^2 \underset{n \to \infty}{\to} K_{3,\delta}$,

(ii) $\sum_{i=1}^{[n\delta]} (X_{in} - \overline{X}_{n,\delta})^2 / [n\delta] \xrightarrow{\text{P}} K_{3,\delta}$,

(iii) $t_{n,\delta}^2 - \sum_{i=1}^{[n\delta]} \left(X_{in} - \bar{X}_{n,\delta} \right)^2 / [n\delta] = \mathrm{O_p}\left(\frac{1}{\ln n} \right),$

(iv) $E(T_{n,\delta}) = a_{n,\delta} = \mathrm{o}(\ln n).$

Now, *provided that* $T_{n,\delta} - E(T_{n,\delta})$ has a limit distribution, L_δ, (i)–(iv) and (2.3) imply that

$$[n\delta](1 - r^2(\boldsymbol{X}, \boldsymbol{H}, \delta)) - a_{n,\delta}/t_{n,\delta}^2 \xrightarrow{\mathrm{D}} L_\delta / K_{3,\delta}. \tag{2.5}$$

Since $a_{n,\delta}/t_{n,\delta}^2 = \mathrm{o}(n^{1/2})$,

$$n^{1/2}(1 - r(\boldsymbol{X}, \boldsymbol{H}, \delta))(1 + r(\boldsymbol{X}, \boldsymbol{H}, \delta)) \xrightarrow{\mathrm{P}} 0.$$

So, since $r(\boldsymbol{X}, \boldsymbol{H}, \delta) \geqslant 0$, $n^{1/2}(1 - r(\boldsymbol{X}, \boldsymbol{H}, \delta)) \xrightarrow{\mathrm{P}} 0$, and we can apply a Slutsky result to (2.5) to obtain

$$2[n\delta](1 - r(\boldsymbol{X}, \boldsymbol{H}, \delta)) - a_{n,\delta}/t_{n,\delta}^2 \xrightarrow{\mathrm{D}} L_\delta / K_{3,\delta}. \tag{2.6}$$

Thus, to complete the proof of the theorem we need to verify that

$$T_{n,\delta} - E(T_{n,\delta}) \xrightarrow{\mathrm{D}} Y_\delta. \tag{2.7}$$

We do so in three main steps.

(1) Develop approximations $c_{kln,\delta}^*$, $d_{kln,\delta}^*$, $e_{kln,\delta}^*$ to $c_{kln,\delta}$, $d_{kln,\delta}$, $e_{kln,\delta}$.

(2) Show that

$$T_{n,\delta}^* - E(T_{n,\delta}^*) \xrightarrow{\mathrm{D}} Y_\delta$$

where

$$T_{n,\delta}^* = \sum_{k=1}^{n+1} \sum_{l=1}^{n+1} \left(c_{kln,\delta}^* - d_{kln,\delta}^* - e_{kln,\delta}^* \right) \frac{Z_k}{(n+1)^{1/2}} \frac{Z_l}{(n+1)^{1/2}}. \tag{2.8}$$

(3) Show that

$$E\left[\left(T_{n,\delta}^* - E(T_{n,\delta}^*) - \left[T_{n,\delta} - E(T_{n,\delta}) \right] \right)^2 \right] \to 0 \quad \text{as } n \to \infty. \tag{2.9}$$

The approximations $c_{kln,\delta}^*$, $d_{kln,\delta}^*$, $e_{kln,\delta}^*$. It is clear from (2.4) that $c_{kln,\delta}$ behaves like $c_\delta(k/(n+1), l/(n+1))$ where

$$c_\delta(x, y) = \int_0^\delta \frac{\psi(x, z)\psi(y, z)}{\phi^2(H(z))} \, \mathrm{d}z. \tag{2.10}$$

It is possible to express $c_\delta(x, y)$ in an alternate form. To see this, consider

$$c_1(x, y) = \int_0^1 \frac{\psi(x, z)\psi(y, z)}{\phi^2(H(z))} \, \mathrm{d}z \tag{2.11}$$

which arises in the complete sample case considered by DeWet and Venter (1973). Because $c_1(x, y)$ is symmetric in x, y and (by properties of the normal distribution) square integrable on $[0, 1]^2$, DeWet and Venter were able to express $c_1(x, y)$ as the quadratic limit of $\sum_{m=1}^\infty \lambda_m f_m(x) f_m(y)$ where the λ_m are the eigenvalues and the $f_m(x)$ are the normalized eigenfunctions of the kernel $c_1(x, y)$. In particular, they showed that

$$\lambda_m = \frac{1}{m} \quad \text{and} \quad f_m(x) = \frac{1}{(2^m M!)^{1/2}} h_m\left(H(x)/2^{1/2} \right)$$

for $m = 0, 1, \ldots,$ where $h_m(\,)$ is the mth Hermite polynomial (see Rainville, 1960) and $H(\,)$ is the inverse of the standard normal distribution function.

Using the complete sample result (2.11), we have

$$c_\delta(x, y) = c_1(x, y) - \int_\delta^1 \frac{\psi(x, z)\psi(y, z)}{\phi^2(H(z))} \mathrm{d}z. \tag{2.12}$$

Thus:

(i) For $0 < x,\ y \leqslant \delta < 1$ and $\delta < z < 1$, we have $\psi(x, z) = \psi(y, z) = (1 - z)$ so

$$c_\delta(x, y) = c_1(x, y) - \int_\delta^1 \frac{(1-z)^2}{\phi^2(H(z))}\,\mathrm{d}z = \sum_{m=1}^{\infty} \frac{1}{m} f_m(x) f_m(y) - K_{c,\delta} \tag{2.13}$$

where $K_{c,\delta}$ is defined in (1.6). The last equality is justified by the fact that $c_1(x, y)$ is also the pointwise limit of its expansion (see Verrill, 1981, Appendix E).

(ii) For $0 < y \leqslant \delta < 1$, $\delta < x < 1$ and $0 < z < \delta$, we have $\psi(x, z) = -z = \psi(\delta, z)$ so

$$c_\delta(x, y) = c_\delta(\delta, y) = \sum_{m=1}^{\infty} \frac{1}{m} f_m(\delta) f_m(y) - K_{c,\delta}. \tag{2.14}$$

(iii) For $0 < x \leqslant \delta < 1$, $\delta < y < 1$,

$$c_\delta(x, y) = \sum_{m=1}^{\infty} \frac{1}{m} f_m(x) f_m(\delta) - K_{c,\delta}. \tag{2.15}$$

(iv) For $\delta < x,\ y < 1$,

$$c_\delta(x, y) = c_\delta(\delta, \delta) = \sum_{m=1}^{\infty} \frac{1}{m} f_m(\delta)^2 - K_{c,\delta}. \tag{2.16}$$

In view of the comment above (2.10) and results (2.13)–(2.16), we approximate $c_{kln,\delta}$ by

$$c^*_{kln,\delta} = \sum_{m=1}^{M_n} \frac{1}{m} g_m\left(\frac{k}{n+1}\right) g_m\left(\frac{l}{n+1}\right) - K_{c,\delta} \tag{2.17}$$

where $g_m(\,)$ is defined by (2.1) and $M_n \to \infty$ as $n \to \infty$.

Similarly, we approximate $d_{kln,\delta}$ by

$$d^*_{kln,\delta} = \frac{1}{\delta}\left(g_1\left(\frac{k}{n+1}\right) + K_{d,\delta}\right)\left(g_1\left(\frac{l}{n+1}\right) + K_{d,\delta}\right) \tag{2.18}$$

and $e_{kln,\delta}$ by

$$e^*_{kln,\delta} = \frac{1}{\delta K_{3,\delta}}\left(\frac{1}{2^{1/2}} g_2\left(\frac{k}{n+1}\right) + K_{e,\delta} - K_{1,\delta}\left(g_1\left(\frac{k}{n+1}\right) + K_{d,\delta}\right)\right)$$
$$\cdot\left(\frac{1}{2^{1/2}} g_2\left(\frac{l}{n+1}\right) + K_{e,\delta} - K_{1,\delta}\left(g_1\left(\frac{l}{n+1}\right) + K_{d,\delta}\right)\right) \tag{2.19}$$

respectively, where $g_1(\,)$, $g_2(\,)$ are defined by (2.1) and $K_{1,\delta}$, $K_{3,\delta}$, $K_{d,\delta}$, and $K_{e,\delta}$ are defined in (1.6).

A straightforward but tedious examination of several cases (see Verrill, 1981, Lemmas 3.6–3.8) establishes

$$\sum_{k=1}^{n+1}\sum_{l=1}^{n+1}(c^*_{kln,\delta}-c_{kln,\delta})^2/(n+1)^2 \underset{n\to\infty}{\to} 0,$$
$$\sum_{k=1}^{n+1}\sum_{l=1}^{n+1}(d^*_{kln,\delta}-d_{kln,\delta})^2/(n+1)^2 \underset{n\to\infty}{\to} 0, \tag{2.20}$$
$$\sum_{k=1}^{n+1}\sum_{l=1}^{n+1}(e^*_{kln,\delta}-e_{kln,\delta})^2/(n+1)^2 \underset{n\to\infty}{\to} 0.$$

The asymptotic distribution of $T^*_{n,\delta}$. From the definitions (2.8) and (2.17)–(2.19), we have

$$\begin{aligned}
T^*_{n,\delta} = {} & \sum_{k=1}^{n+1}\sum_{l=1}^{n+1}\left(\sum_{m=1}^{M_n}\frac{1}{m}g_m\left(\frac{k}{n+1}\right)g_m\left(\frac{l}{n+1}\right)-K_{c,\delta}\right)\frac{Z_kZ_l}{n+1}\\
& -\sum_{k=1}^{n+1}\sum_{l=1}^{n+1}\frac{1}{\delta}\left(g_1\left(\frac{k}{n+1}\right)+K_{d,\delta}\right)\left(g_1\left(\frac{l}{n+1}\right)+K_{d,\delta}\right)\frac{Z_kZ_l}{n+1}\\
& -\sum_{k=1}^{n+1}\sum_{l=1}^{n+1}\frac{1}{\delta K_{3,\delta}}\left(\frac{1}{2^{1/2}}g_2\left(\frac{k}{n+1}\right)-K_{1,\delta}\left(g_1\left(\frac{k}{n+1}\right)+K_{d,\delta}\right)+K_{e,\delta}\right)\\
& \qquad\cdot\left(\frac{1}{2^{1/2}}g_2\left(\frac{l}{n+1}\right)-K_{1,\delta}\left(g_1\left(\frac{l}{n+1}\right)+K_{d,\delta}\right)+K_{e,\delta}\right)\frac{Z_kZ_l}{n+1}\\
= {} & \sum_{k=0}^{n+1}\sum_{l=0}^{n+1}\left(\sum_{m=1}^{M_n}\frac{1}{m}g_m\left(\frac{k}{n+1}\right)g_m\left(\frac{l}{n+1}\right)\right.\\
& \qquad\left.+\sum_{0\leqslant j\leqslant i\leqslant 2}J_{ij,\delta}g_i\left(\frac{k}{n+1}\right)g_j\left(\frac{l}{n+1}\right)\right)\frac{Z_kZ_l}{n+1}.
\end{aligned} \tag{2.21}$$

Now the properties of the Hermite polynomials and the normal inverse $H(\ )$ imply that the $g_m(\ )$'s are well behaved. We state this result in the form of a lemma (cf. Verrill, 1981, Appendix G).

Lemma 1. *For any fixed* $l, m \in \{1, 2, \dots\}$,

$$\max_{0\leqslant i\leqslant n+1}\frac{|g_m(i/(n+1))|}{(n+1)^{1/2}} \underset{n\to\infty}{\to} 0,$$
$$\sum_{i=0}^{n+1}g_l\left(\frac{i}{n+1}\right)g_m\left(\frac{i}{n+1}\right)/(n+1) \underset{n\to\infty}{\to} \int_0^1 g_l(x)g_m(x)\,\mathrm{d}x,$$
$$\sum_{l=n_1+1}^{n_2}\sum_{m=n_1+1}^{n_2}\frac{1}{lm}\left(\int_0^1 g_l(x)g_m(x)\,\mathrm{d}x\right)^2 \underset{n_1,n_2\to\infty}{\to} 0. \quad \square$$

By equation (2.21), Lemma 1, and Corollary 6 of Verrill and Johnson (1988b), we have

$$T^*_{n,\delta}-E(T^*_{n,\delta}) \overset{\mathrm{D}}{\to} Y_\delta. \tag{2.22}$$

Proof of Theorem 1 (continued). Given (2.7) and (2.22), it is clear that the theorem will be established if

$$\Delta_n = E\left(\left[T_{n,\delta}^* - E(T_{n,\delta}^*) - T_{n,\delta} + E(T_{n,\delta})\right]^2\right) \underset{n\to\infty}{\to} 0. \tag{2.23}$$

To establish (2.23), let

$$a_{kln} = \frac{c_{kln,\delta}^* - c_{kln,\delta} - (d_{kln,\delta}^* - d_{kln,\delta}) - (e_{kln,\delta}^* - e_{kln,\delta})}{n+1} \tag{2.24}$$

By the definitions of $T_{n,\delta}$ and $T_{n,\delta}^*$, we have

$$\begin{aligned}\Delta_n &= E\left(\left[T_{n,\delta}^* - E(T_{n,\delta}^*) - T_{n,\delta} + E(T_{n,\delta})\right]^2\right)\\ &= E\left(\left[T_{n,\delta}^* - T_{n,\delta}\right]^2\right) - E^2(T_{n,\delta}^* - T_{n,\delta})\\ &= E\left(\left[\sum_{k=1}^{n+1}\sum_{l=1}^{n+1} a_{kln} Z_k Z_l\right]^2\right) - \left(\sum_{k=1}^{n+1} a_{kkn}\right)^2\\ &= \sum_{k=1}^{n+1} a_{kkn}^2(\mu_4 - 3) + 2\sum_{k=1}^{n+1}\sum_{l=1}^{n+1} a_{kln}^2 \end{aligned} \tag{2.25}$$

where $\mu_4 = E(Z_k^4)$. By the limits (2.20) and the generalized Minkowski inequality,

$$\sum_{k=1}^{n+1}\sum_{l=1}^{n+1} a_{kln}^2 \underset{n\to\infty}{\to} 0. \tag{2.26}$$

Results (2.25) and (2.26) yield (2.23) which completes the proof. □

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