FITTING STATISTICAL DISTRIBUTION MODELS TO MOE AND MOR IN MILL-RUN SPRUCE AND RED PINE LUMBER POPULATIONS[1]

*Guangmei Cao Anderson**

Graduate Research Assistant
E-mail: gc817@msstate.edu

Frank C. Owens†

Assistant Professor
Mississippi State University
Mississippi State, MS 39762-5724
E-mail: fco7@msstate.edu

Steve P. Verrill†

Mathematical Statistician
E-mail: steve.p.verrill@usda.gov

Robert J. Ross

Supervisory Research General Engineer
USDA Forest Products Laboratory
Madison, WI 53726
E-mail: robert.j.ross@usda.gov

Rubin Shmulsky†

Professor and Department Head
Mississippi State University
Mississippi State, MS 39762-5724
E-mail: rs26@msstate.edu

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Abstract. It has been mathematically demonstrated that the distribution of MOR in a graded lumber subpopulation does not have the same theoretical form as the distribution of the mill-run population from which the subpopulation is drawn. However, the distributional form of the graded lumber subpopulation does depend heavily on the distributional form of the full mill-run population, and thus, it is important to characterize the distributions of full mill-run lumber populations. Previous studies presented evidence suggesting that commonly used distributions such as normal, lognormal, and Weibull distributions might not be suitable for modeling mill-run MOE and MOR; rather, nontraditional distributions such as skew normal and mixed normal seem to be more appropriate models for the MOE and MOR of mill-run populations across mills and time. Previous studies of this kind have been carried out using only southern pine (*Pinus* spp.) lumber. In this study, we extend this work by investigating whether the distributional forms found to adequately fit southern pine mill-run lumber populations also adequately fit other species (or species groups). The objective of this study was to identify statistical models that fit MOE and MOR distributions in mill-run spruce (*Picea* spp.) and red pine (*Pinus resinosa*) lumber populations. Mill-run samples of 200 spruce 2 × 4 specimens and 200 red pine 2 × 4 specimens (for a total of 400 test pieces) were collected, and the MOE and MOR for each specimen were assessed. Various distributions were fit to the MOE and MOR mill-run data and evaluated for goodness of fit. In addition to further demonstrating that traditional distributions such as normal, lognormal, and Weibull may not be adequate to model mill-run MOE and MOR populations, the results suggested that mixed normal and skew normal distributions might perform well across species.

* Corresponding author
† SWST member

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INTRODUCTION

The MOE of graded lumber is often modeled as a normal distribution, and the MOR of graded lumber as a normal, lognormal, or Weibull distribution (Green and Evans 1987; Evans et al 1997; ASTM 2017a, 2017b). However, Verrill et al (2012, 2015) demonstrated mathematically that the distribution of MOR in a graded subpopulation will not have the same theoretical form as the distribution of the MOR in the full, ungraded (or "mill-run") population from which the subpopulation is drawn. Instead, the distributional form of the MOR in graded lumber will be pseudo-truncated, exhibiting thinned tails (Verrill et al 2012, 2015).

Verrill et al (2013, 2014, 2019, 2020a) established empirically that the MOR distributions of visual grades of lumber are not two-parameter Weibulls and do indeed display pseudo-truncated behavior.

Because the exact form of a pseudo-truncated MOR distribution for graded lumber depends heavily on the mill-run MOR population from which the graded subpopulation is drawn, it is important to characterize the distributions of mill-run populations. Based on research that studied the bending properties of eight mill-run southern pine (*Pinus* spp.) lumber populations from different mills and different times of the year, Owens et al (2020) presented evidence suggesting that normal, lognormal, and Weibull distributions might not be suitable for modeling mill-run MOE and MOR, and that nontraditional distributions such as skew normal and mixed normal seem to be more adequate models for the MOE and MOR of those mill-run populations.

Previously, studies of this kind have been carried out using only southern pine lumber. In this study, we extend this work by investigating whether the distributional forms found to adequately fit southern pine mill-run lumber populations also adequately fit other species (or species groups). The objective of this study was to identify statistical models that adequately fit MOE and MOR distributions in mill-run red pine (*Pinus resinosa*) and spruce (*Picea* spp.) lumber populations.

MATERIALS AND METHODS

Sampling

Two mill-run samplings were acquired for this research. One was sampled from spruce production at a New Hampshire mill, and the other was sampled from red pine production at a Minnesota mill. Each sampling consisted of 200 pieces of rough, kiln-dried 2 × 4 lumber. The nominal length was 8 feet (243.84 cm).

For each sampling, from weekly kiln production, a kiln package was randomly selected. After the top course of lumber was removed, the subsequent 200 pieces of lumber were sampled. A detailed sampling scheme can be found in Owens et al (2018, 2019). All materials were pulled before grades were assigned. The samplings include qualities from the "best" to the "worst," even pieces that might not make grade and otherwise be discarded.

All samples were transported to (name of university redacted for review) and planed to average dimensions of 1.5 inches by 3.5 inches (3.81 × 8.89 cm). Although specimens were pulled as mill-run materials, all the samples were graded by a Northeastern Lumber Manufacturers Association (NELMA, Cumberland Center, ME)–certified grader for additional data in case of further analysis.

Testing

For each specimen, elasticity (static MOE and dynamic MOE) and strength (MOR) properties were assessed. Nondestructive tests were performed to estimate two forms of dynamic MOE (transverse and longitudinal), and a static bending test per ASTM D 198-15 (a destructive test) was performed to measure the static MOE and MOR.

Metriguard's E-Computer Model 340 (Metriguard, Inc., Pullman, WA, www.metriguard.com) was used to estimate the dynamic MOE by measuring transverse vibration. Each specimen was placed flatwise on two tripods supporting the test piece at each end. One tripod was topped with a transducer connected to a computer to capture the oscillation introduced by a slight tap at the middle of the board. After sensing the vibration and measuring the weight, the E-computer calculated the dynamic MOE by Eq 1:

$$E = \left(f^2 W S^3\right) / (CIg), \quad (1)$$

where E is dynamic MOE, f is the resonant frequency, W is the weight, S is the span, C is a constant, I is the moment of inertia, and g is the acceleration due to gravity (Ross 2015).

The other form of dynamic MOE was estimated by measuring longitudinal acoustic velocity using Fibre-gen's Director HM200 (Fibre-gen Limited, Christchurch, New Zealand, www.fibre-gen.com). Each test piece was placed flatwise on two sawhorses with the sensor of the Fibre-gen device held firmly against one end of the board. A hammer was used to tap at the same end and initiate an acoustic wave that traveled back and forth longitudinally. The acoustic velocity value measured by the sensor was used in Eq 2 to calculate the MOE value.

$$E = \rho V^2, \quad (2)$$

where E is dynamic MOE, ρ is density, and V is velocity.

Static MOE and MOR were measured by a third-point static bending test performed on an Instron universal testing machine per ASTM D 198-15 (ASTM 2015) with a span-to-depth ratio of 17:1 (Fig 1). The test span was thus 59.5 inches (151.13 cm). This 59.5-inch span was then randomly positioned, lengthwise, to each test specimen's approximate 96-inch length. Each test specimen was randomly determined and marked within the eight-foot-long board before the test. The test piece was placed edgewise in the fixture. A load was applied until full rupture, and the deflection was measured by an extensometer placed under the bottom edge of the midspan. Before any statistical analysis, MOR and three measures of MOE were adjusted to a common MC of 15% per ASTM D 1990-16 (ASTM 2016). The average MC of the spruce samples was 15.1% (SD = 0.96), and the average MC of the red pine samples was 11.5% (SD = 0.80).

One specimen of spruce broke before the bending test, and that record was removed listwise. Also, Metriguard's E-computer was unable to obtain the dynamic MOE value of one specimen of spruce. Fibre-gen's Director HM200 was unable to obtain the dynamic MOE values of two specimens of spruce and five specimens of red pine. Accordingly, the missing data were removed pairwise.

Statistical Methods

In total, six distributions were selected as candidate distributions to fit to the MOR and MOE data so that they could be evaluated for goodness of fit. Based on past studies (Galligan et al 1986; Green and Evans 1987; Evans et al 1997; ASTM 2017a, 2017b), four commonly used distributions—normal, lognormal, two-parameter Weibull, and three-parameter Weibull—were selected. In addition, two nontraditional distributions—skew normal, and mixed normal—were selected based on previous research (Verrill et al 2017; Owens et al 2018, 2019, 2020). The probability density function for each distribution can be found in the Appendix section.

R (R Core Team 2018) was used to perform normal and lognormal fits. Fortran programs written by the authors were used to perform the two-parameter Weibull, three-parameter Weibull, skew normal, and mixed normal fits. To assess goodness of fit, the nortest package (Gross and Ligges 2015) in *R* was used to perform Cramér–von Mises (CVM) and Anderson–Darling tests for normal and lognormal distributions. The *R* shapiro.test command was used to perform Shapiro–Wilk tests for normal and lognormal distributions. The EWGoF package (Krit 2017) in *R* was used to perform CVM and Anderson–Darling tests for the two-parameter Weibull distribution. Fortran programs using a parametric bootstrap simulation were used to perform CVM tests for three-parameter Weibull, skew normal, and mixed normal distributions.

Figure 1. Third-point static bending test fixture per ASTM D 198-15.

RESULTS

The results of the goodness-of-fit tests for spruce and red pine are presented in Tables 1 and 2. The histograms and probability plots for each combination of four variables (static MOE, E-computer E, Director E, and MOR) and six distributions (normal, lognormal, two-parameter Weibull, three-parameter Weibull, skew normal, and mixed normal) for both species can be found at https://www1.fpl.fs.fed.us/spruce_redpine_plots.html.

Table 1. GOF *p*-values for spruce.

			Distribution					
Property	*N*	GOF test	Normal	Lognormal	Two-par Weibull	Three-par Weibull	Skew normal	Mixed normal
Static MOE	199	Shapiro–Wilk	**0.46**	<0.0001	—	—	—	—
		CVM	**0.31**	<0.0001	**0.87**	—	—	—
		Anderson–Darling	**0.33**	<0.0001	**0.88**	—	—	—
		CVM simulation[a]	—	—	—	**0.775**	**0.365**	**0.604**
E-computer E	198	Shapiro–Wilk	**0.26**	<0.0001	—	—	—	—
		CVM	**0.25**	<0.0001	**0.98**	—	—	—
		Anderson–Darling	**0.27**	<0.0001	**0.98**	—	—	—
		CVM simulation	—	—	—	**0.957**	**0.999**	**0.740**
Director E	197	Shapiro–Wilk	**0.67**	<0.0001	—	—	—	—
		CVM	**0.56**	<0.0001	**0.43**	—	—	—
		Anderson–Darling	**0.57**	<0.0001	**0.34**	—	—	—
		CVM simulation	—	—	—	**0.465**	**0.785**	**0.873**
MOR	199	Shapiro–Wilk	**0.20**	<0.0001	—	—	—	—
		CVM	**0.68**	<0.0001	**0.73**	—	—	—
		Anderson–Darling	**0.50**	<0.0001	**0.53**	—	—	—
		CVM simulation	—	—	—	**0.259**	**0.539**	**0.237**

CVM, Cramér–von Mises; GOF, goodness of fit; par, parameter; E-computer E, dynamic MOE as tested with the E-computer device; Director E, dynamic MOE as tested with the Director HM200 device; "—," the test was not performed. Bold values indicate that a test failed to reject a distribution at a 0.05 significance level.
[a] In cases where critical values for the CVM test were not available in D'Agostino and Stephens (1986), they were determined by simulation.

Table 2. GOF *p*-values for red pine.

Property	*N*	GOF test	Distribution					
			Normal	Lognormal	Two-par Weibull	Three-par Weibull	Skew normal	Mixed normal
Static MOE	200	Shapiro–Wilk	<0.0001	**0.77**	—	—	—	—
		CVM	<0.0001	**0.67**	<0.0001	—	—	—
		Anderson–Darling	<0.0001	**0.73**	<0.0001	—	—	—
		CVM simulation[a]	—	—	—	**0.059**	**0.351**	**0.249**
E-computer E	200	Shapiro–Wilk	<0.0001	**0.47**	—	—	—	—
		CVM	<0.0001	**0.49**	<0.0001	—	—	—
		Anderson–Darling	<0.0001	**0.52**	<0.0001	—	—	—
		CVM simulation	—	—	—	0.044	**0.549**	**0.921**
Director E	195	Shapiro–Wilk	<0.0001	**0.43**	—	—	—	—
		CVM	<0.0001	**0.55**	<0.0001	—	—	—
		Anderson–Darling	<0.0001	**0.48**	<0.0001	—	—	—
		CVM simulation	—	—	—	0.001	**0.247**	**0.723**
MOR	200	Shapiro–Wilk	<0.0001	**0.05**	—	—	—	—
		CVM	<0.0001	**0.09**	<0.0001	—	—	—
		Anderson–Darling	<0.0001	**0.09**	<0.0001	—	—	—
		CVM simulation	—	—	—	<0.001	0.007	**0.974**

GOF, goodness of fit; par, parameter; E-computer E, dynamic MOE as tested with the E-computer device; Director E, dynamic MOE as tested with the Director HM200 device; CVM, Cramér–von Mises; "—," the test was not performed. Bold values indicate that a test failed to reject a distribution at a 0.05 significance level.
[a] In cases where critical values for the CVM test were not available in D'Agostino and Stephens (1986), they were determined by simulation.

Table 3 shows the number of times that a distribution was rejected by the goodness-of-fit test at a 0.05 significance level. (A value of 0 indicates that it failed to reject, and a value of 1 indicates that it was rejected.) The bottom row shows the total number of times the distribution was rejected across species and properties. The value can range from 0 to 8. Lower numbers suggest that the distribution might be a good model for strength and stiffness across these species. Probability plots for mixed normal and skew normal distributional fits to bending properties in spruce and red pine appear in Figs 2 and 3.

DISCUSSION

For the southern pine data in Owens et al (2020), the normal, lognormal, two-parameter Weibull, and three-parameter Weibull distributions showed differences in fit quality *among mills and between seasons*. In the current study of spruce and red pine, these four distributions showed differences in fit quality *between species*. For example, while normal and two-parameter Weibull showed adequate fits across all properties in the spruce data, they yielded poor fits across all properties in the red pine data. On the other hand, while lognormal showed adequate fits across all properties in the red pine data, it yielded poor fits across all properties in the spruce data. Similarly, three-parameter Weibull exhibited adequate fit for all properties in spruce but poor fit for all properties, except static MOE in red pine. When the comparison also includes the results of the goodness-of fit-tests for southern pine reported in Owens et al

Table 3. Goodness-of-fit test summary rejection score card.

Species	Property	Normal	Lognormal	Two-par Weibull	Three-par Weibull	Skew normal	Mixed normal
Spruce	Static MOE	0	1	0	0	0	0
	E-computer E	0	1	0	0	0	0
	Director E	0	1	0	0	0	0
	MOR	0	1	0	0	0	0
Red pine	Static MOE	1	0	1	0	0	0
	E-computer E	1	0	1	1	0	0
	Director E	1	0	1	1	0	0
	MOR	1	0	1	1	1	0
Total		4	4	4	3	1	0

par, parameter; E-computer E, dynamic MOE as tested with E-computer device; Director E, dynamic MOE as tested with Director HM200 device. The numbers in the table indicate the number for which a distribution was rejected by a goodness-of-fit test at a 0.05 significance level for each of the four properties (static MOE, E-computer E, Director E, and MOR). These numbers (except for total) can be either 0 (failed to reject) or 1 (rejected).

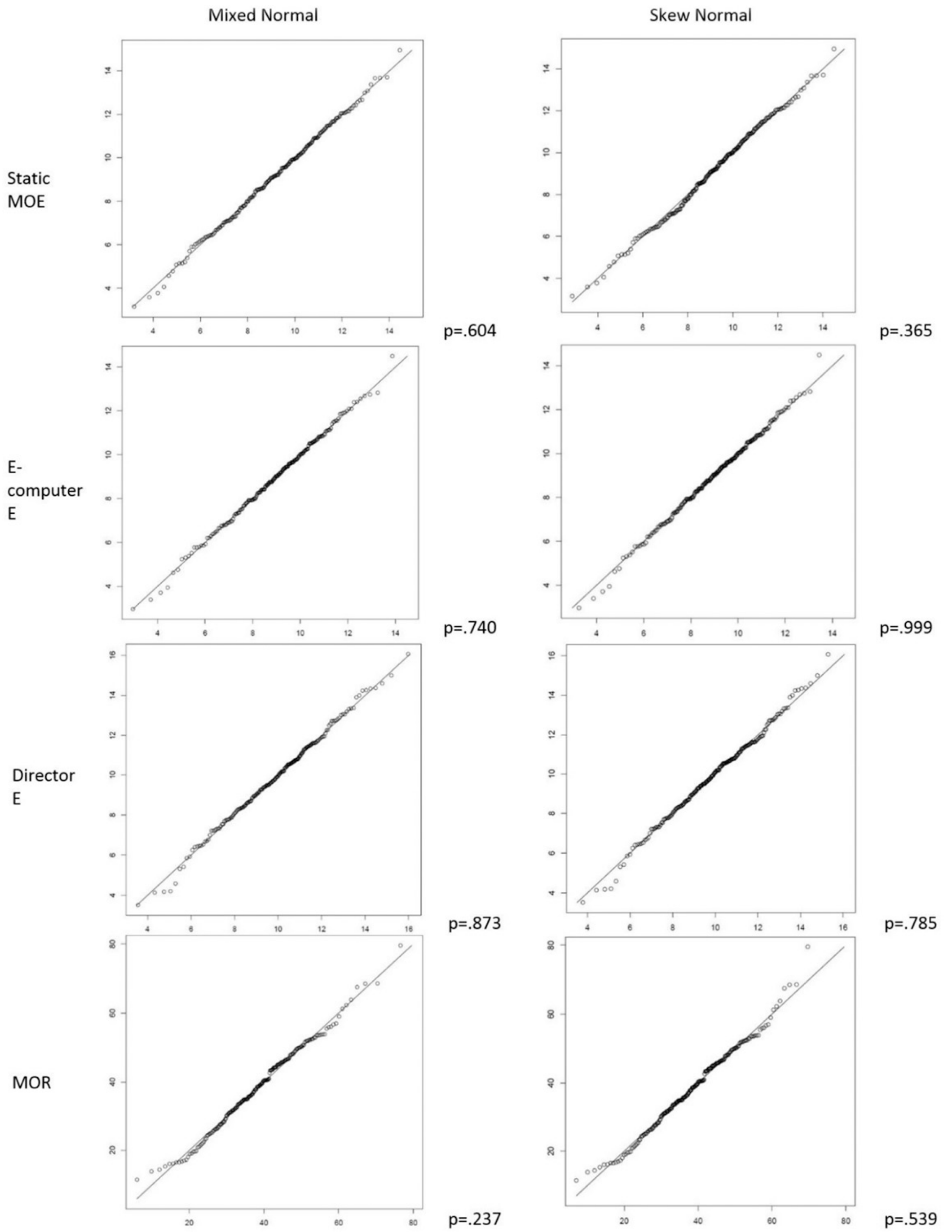

Figure 2. Probability plots for skew normal and mixed normal distributional fits to bending properties in spruce. For each plot, X and Y axes are "ordered expected values" and "ordered observed values," respectively. The *p*-values are from Cramér–von Mises simulation tests.

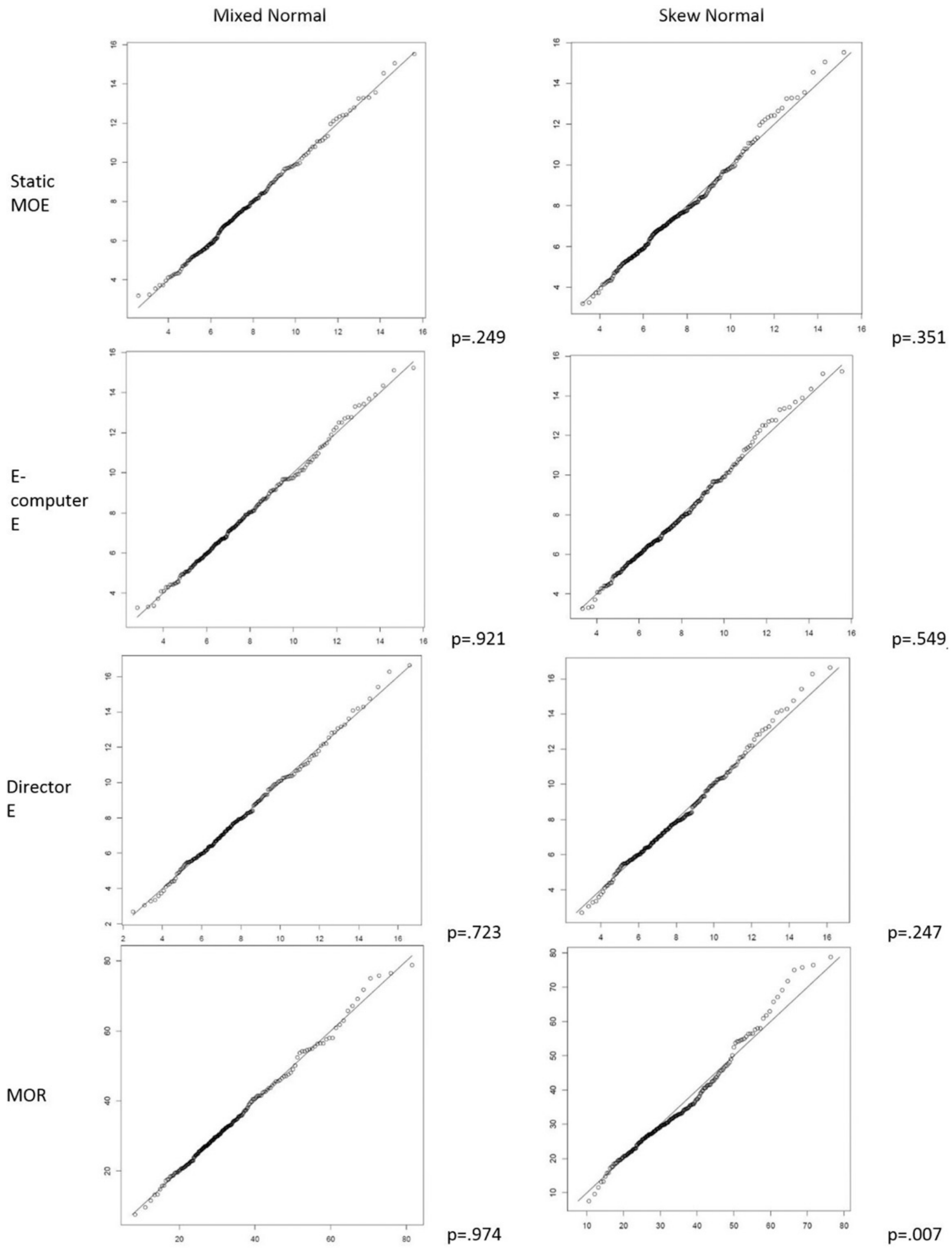

Figure 3. Probability plots for skew normal and mixed normal distributional fits to bending properties in red pine. For each plot, X and Y axes are "ordered expected values" and "ordered observed values," respectively. The *p*-values are from Cramér–von Mises simulation tests.

(2018, 2019, 2020), these inconsistencies in fit quality across species become even more apparent.

Based on tallies in Table 3, if one were to rank these six distribution models from best to worst according to the number of properties for which they seemed an adequate fit across both the spruce and red pine data, the order would be 1) mixed normal, 2) skew normal, 3) three-parameter Weibull, and 4) normal, lognormal, and two-parameter Weibull (in a three-way tie). Similar to the results reported in Owens et al (2018, 2019, 2020), skew normal and mixed normal performed well across all properties in both spruce and red pine, as illustrated in Figs 2 and 3.

The results of all the mill-run studies thus far suggest that these two distributions might perform better across properties, mills, time, and even species than the traditional normal, lognormal, and Weibull distributions. We note however that recent work suggests that even if we can find a class of models that yields good fits to mill-run strength and stiffness distributions, the *particular fits* (the fitted parameters) can vary significantly from mill to mill and time to time.

For example, Anderson et al (2019) established that the means and standard deviations of mill-run property distributions could vary with time and that these variations could yield significant differences in the percentiles of the strength distributions of the corresponding grades produced by the mills. Verrill et al (2020b) demonstrated that observed changes from mill to mill and time to time sometimes yield large changes in the probabilities of board breakage at fixed loads for boards from the same nominal "grade."

Thus, although it is important to identify general classes of models that do a good job of fitting mill-run strength and stiffness distributions at a wide variety of mills, if the resulting *specific* fits vary widely from time to time and from mill to mill, scientists and engineers will still have a strong incentive to develop computer models that yield real-time in-line estimates of lumber properties based on measurements of stiffness, specific gravity, knot size and location, slope of grain, and other strength predictors.

CONCLUSION

The objective of this study was to identify statistical models that adequately fit MOE and MOR distributions in mill-run red pine and spruce lumber populations. Mill-run samples of 200 red pine 2 × 4 specimens and 200 spruce 2 × 4 specimens (for a total of 400 test pieces) were collected, and the MOE and MOR for each specimen were assessed. Six distributions were fit to the MOE and MOR mill-run data and evaluated for goodness of fit. The results demonstrated that traditional distributions such as normal, lognormal, and Weibull might not be adequate to model MOE and MOR populations across species. Mixed normal and skew normal performed well across species.

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APPENDIX—PROBABILITY DENSITY FUNCTIONS NORMAL DISTRIBUTION

NORMAL

The normal probability density function is given by

$$f(x;\ \mu,\sigma)=\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(\frac{-(x-\mu)^2}{(2\sigma^2)}\right)$$

for $x\in(-\infty,\infty)$, where μ is the mean and σ is the SD. This distribution is denoted by the notation $N(\mu,\sigma^2)$.

LOGNORMAL

The lognormal probability density function is given by

$$f(x;\ \mu,\sigma)=\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\frac{1}{x}\exp\left(\frac{-(\log(x)-\mu)^2}{(2\sigma^2)}\right)$$

for $x\in(0,\infty)$, where μ is the mean and σ is the SD of the log of the original data.

TWO-PARAMETER WEIBULL

The two-parameter Weibull has probability density function

$$f(w;\gamma,\beta)=\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)$$

for $w\in(0,\infty)$, where β is the shape parameter and γ is the inverse of the scale parameter.

THREE-PARAMETER WEIBULL

The three-parameter Weibull has probability density function

$$f(w;\gamma,\beta,c)=\gamma^\beta\beta(w-c)^{\beta-1}\exp(-\gamma(w-c))^\beta$$

for $w\in(\ c,\infty)$, where β is the shape parameter, γ is the inverse of the scale parameter, and c is the location parameter.

SKEW NORMAL

The skew normal distribution has probability density function

$$f(x;\xi,\omega,\alpha)=\frac{2}{\omega}\times\phi\left(\frac{x-\xi}{\omega}\right)\times\Phi\left(\alpha\left(\frac{x-\xi}{\omega}\right)\right)$$

for $x\in(-\infty,\infty)$, where ϕ denotes the probability density function of a standardized normal, Φ denotes the cumulative distribution function of a standardized normal, and ξ, ω, and α are the parameters of the skew normal.

MIXED NORMAL

Here, the "mixed normal distribution" refers to a mixture of *two* normal distributions. Such a mixture results when specimens are drawn with probability p from a $N(\mu_1,\sigma_1^2)$ distribution and with probability $1-p$ from a $N(\mu_2,\sigma_2^2)$ distribution. The probability density function is given by

$$f(x;\ \mu_1,\sigma_1,p,\ \mu_2,\sigma_2)= p\times\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_1}\exp\left(\frac{-(x-\mu_1)^2}{(2\sigma_1^2)}\right)+(1-p)\times\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_2}\exp\left(\frac{-(x-\mu_2)^2}{(2\sigma_2^2)}\right)$$

for $x\in(-\infty,\ \infty)$.