A Reminder about Potentially Serious Problems with a Type of Blocked ANOVA Analysis

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1 Introduction

There is a type of blocked experiment that has the potential of being poorly designed and/or analyzed. Verrill *et al.* (1993, 1999, 2004) referred to such an experiment as a "predictor sort" experiment. David and Gunnink (1997) spoke of "artificial pairing." In textbooks it is sometimes referred to as a "matched pair" or "matched subjects" design. The associated design process is also sometimes described as "forming blocks via a concomitant variable." In a wood research context, Warren and Madsen (1977) described the specimen allocation procedure as follows:

> One can take steps, however, to ensure that the inherent [initial] strength distributions of test and control samples are reasonably equivalent. Indeed, failure to do so can only throw doubt on the results.
>
> Specifically, then, all the boards in the experiment are ordered from weakest to strongest as nearly as can be judged from their moduli of elasticity, knot size, and slope of grain. To divide the material into J equivalent groups the first J boards, after ordering, are taken and randomly allocated one to each group. This is repeated with the second, third, fourth, etc., sets of J boards. The strength distributions of the resulting groups should then be essentially the same.

Here the response is lumber strength after a treatment, and the predictor/concomitant used to form blocks (of size J) would be some combination of lumber stiffness, knot size, and slope of grain (all of which can be measured non-destructively prior to specimen allocation).

In an agricultural context, the predictor/concomitant variable might be, for example, animal age, initial animal weight, or plot fertility in a previous trial. In a behavioral or educational context, the predictor might be, for example, IQ or performance on a pre-test.

In this paper we will refer to this type of design as a "predictor sort" design (because we sort specimens on the basis of a predictor that is correlated with the response, and then form blocks via collections of specimens with adjacent predictor values). Our theory will be established for the case in which the predictor and the response have a joint bivariate normal distribution.

In his 1999 paper, Verrill cited discussions of this type of experiment in example 3.3 of Cox (1958), section 8.2 of Steel and Torrie (1960), section 5.1 of Kirk (1968), section 13.17 of Finney (1972), example 11.3 of Ostle and Mensing (1975), Chapter 6 of Myers (1979), and example 6.13.1 of Snedecor and Cochran (1989). A more recent sampling of statistical texts found such experiments discussed in Kerlinger and Lee (1999), van Zutphen *et al.* (2001), example 5.1 of Toutenburg (2002), section 4.3 of Ruxton and Colegrave (2006), problem 3.8 of Casella (2008), Cozby and Bates (2011), Tuckman and Harper (2012), and section 8.1 of Kirk (2013).

Among the variables suggested as predictors/concomitants to be used to form blocks were age, reaction time, initial weight, concentration of blood constituent, degree of disease, time since college, IQ, scores on a cognitive ability measure, grade point average, prior school performance, and pretest achievement.

Improperly designed and/or analyzed, predictor sort experiments can be associated with incorrect/inadequate power calculations and sample sizes, incorrect tests of hypotheses, and incorrect confidence intervals. Verrill (1993) and David and Gunnink (1997) focused on potential problems with hypothesis tests given a predictor sort design. Verrill (1999) focused on confidence intervals on means. Verrill *et al.* (2004) focused on confidence intervals on quantiles. Because incorrect predictor sort designs and analyses can have serious adverse effects on decision-making, and because this fact has not yet become common knowledge among statisticians (or at least among textbook authors), in this paper we review the main results in the literature, add a section on multiple comparisons, and present the results from power and confidence interval coverage simulations that emphasize the importance of the proper design and analysis of predictor sort experiments.

In section 2 we focus on tests of hypotheses. In section 3 we discuss confidence intervals on means. In section 4 we discuss confidence intervals on quantiles. In section 5 we discuss Scheffé and Tukey multiple comparison tests and the corresponding simultaneous confidence intervals. And in section 6, we describe web/R programs that we have written to aid in the design and analysis of predictor sort experiments.

Analysis of covariance and maximum likelihood estimation

In this paper we focus on predictor sort designs, and special predictor sort analyses of data obtained from predictor sort experiments. We contrast the performance of the predictor sort analyses with standard anova analyses. However, in our simulations and theoretical work, we also address analysis of covariance and maximum likelihood solutions to the problems. We find that an unmodified analysis of covariance works well for tests of hypotheses, but not for confidence intervals. We further find that a maximum likelihood approach to confidence intervals yields correct asymptotic coverages, but that predictor sort coverages approach nominal coverages more rapidly than do maximum likelihood coverages. Further, a maximum likelihood approach is considerably more complex, and does not readily reveal the $1-\rho^2+\rho^2/J$ asymptotic factor (first discussed in Section 3) that is associated with all three (predictor sort, analysis of covariance, and maximum likelihood) approaches.

2 Hypothesis Tests

We first set some useful notation. Here, for ease of exposition, we will restrict ourselves to the one-factor case. Let Y_{ij} denote the response for the ith block, $i \in \{1,\ldots,I\}$, of the jth treatment, $j \in \{1,\ldots,J\}$. Let ρ denote the correlation between the predictor/concomitant, X, and the response, Y. We assume that X and Y have a joint bivariate normal distribution.

In a non predictor sort case, the probability model for a blocked ANOVA would be

$$Y_{ij} = \mu_{\cdot j} + \mu_{i\cdot} + \sigma_Y \times \epsilon_{ij} \tag{1}$$

where $\mu_{\cdot 1},\ldots,\mu_{\cdot J}$ denote the treatment effects, $\mu_{1\cdot},\ldots,\mu_{I\cdot}$ denote the block effects, and the ϵ's are i.i.d. N(0,1)'s. In a predictor sort case, we have $n = JI$ specimens. To allocate these specimens, we order the X's and randomly assign the J specimens associated with the lowest X values, to the first block, the specimens associated with the next J lowest values to the next block, and so on. In this case, the correct probability model is

$$Y_{ij} = \mu_j + \sigma_Y \left(\rho\left(X_{k(i,j),n} - \mu_X\right)/\sigma_X + \sqrt{1-\rho^2}P_{ij}\right) \tag{2}$$

where $\mu_1, \ldots, \mu_J$ denote the treatment effects; for a fixed i, the J $k(i,j)$'s are a randomization of the elements of $\{(i-1)J+1, \ldots, iJ\}$; $X_{l,n}$ denotes the lth order statistic among the X's; and the P_{ij}'s are i.i.d. N(0,1)'s that are independent of the X's.

The differences between models (1) and (2) (and, in particular, the fact that $X_{k(i,j_1),n} - X_{k(i,j_2),n}$ tends to be smaller than an arbitrary $X_1 - X_2$ and yet not equal to 0) are the source of both the advantages and the problems associated with predictor sort experiments. (More detailed heuristic discussions are provided in Verrill (1993), Verrill and Green (1996), Verrill (1999), and Verrill *et al.* (2004).)

Verrill (1993) proved the following theorem on hypothesis testing following a predictor sort allocation.

Theorem 1

Assume that the predictor variable and the variable of interest have a joint bivariate normal distribution with correlation ρ. Let the allocation of samples be as described in Section 1. (For the multi-factor case, enough adjacent (in predictor values) experimental units are chosen at a time to provide one additional observation for each cell.) Then, for a factor with J levels, for $0 \leq \rho < 1$, the asymptotic distribution of the ANOVA test statistic that treats the groups of adjacent (in predictor values) experimental units as a block is $\chi^2_{J-1}/(J-1)$. The asymptotic distribution of the ANOVA test statistic that ignores the block structure generated by these groups is $(1-\rho^2)\chi^2_{J-1}/(J-1)$.

Proof

See Verrill (1993) for a 1-factor proof, and Appendix G of the current paper for a multi-factor proof.

Because of this asymptotic behavior, if we analyze a predictor sort experiment as a blocked ANOVA (where the blocks are formed of specimens with similar predictor/concomitant values — "matched subjects") and I is sufficiently large, the nominal size of the test will be approximately equal to the true size. (Actually, for very large ρ values, the true size is reduced from the nominal size even for fairly large samples. See the web version of Table 1 referenced below.) However, if we ignore the blocks in our analysis, the actual size can be *much* lower than nominal size and power will suffer significantly. (We essentially end up comparing $(1-\rho^2)\chi^2_{J-1}$ random variables with χ^2_{J-1} critical values.)

To explore these effects we have performed a large power simulation of the 1-factor case. For all combinations of X,Y correlations 0.0, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, and 0.99; number of treatments, J, equal to 2, 3, 5, 7, 9, 11, and 20; sample sizes, I, equal to 3, 5, 10, 20, and 40; and 21 noncentrality parameters, we performed 40,000 trials. We created two versions of each of the resulting data sets. One version was created by allocating the specimens in a data set to the J treatment conditions via a standard randomization. The second version was created by allocating the specimens in a data set to the J treatment conditions via a predictor sort.

We then performed performed seven hypothesis tests on the data sets, and two "theoretical" power calculations:

1. A standard 1-way analysis of variance on the non predictor sort version of the data set. (The estimated power is reported in column 5 of Table 1.)

2. A standard (and thus incorrect) 1-way analysis of variance on the predictor sort version of the data set. (The estimated power is reported in column 6 of Table 1.)

3. A *corrected* 1-way analysis of variance on the predictor sort version of the data set. The corrected F statistic is the standard 1-way statistic divided by $1-\hat{\rho}^2$. (The estimated power is reported in column 7 of Table 1.)

4. A second *corrected* 1-way analysis of variance on the predictor sort version of the data set. The corrected F statistic is the standard 1-way statistic divided by $1-\rho^2_{\text{true}}$. (The estimated power is reported in column 8 of Table 1.)

5. A 2-way analysis of variance on the predictor sort version of the data set. (The blocks are formed by specimens with adjacent [randomized within the block] values of the predictor.) (The estimated power is reported in column 9 of Table 1.)

6. An analysis of covariance on the non predictor sort version of the data set. (The estimated power is reported in column 10 of Table 1.)

7. An analysis of covariance on the predictor sort version of the data set. (The estimated power is reported in column 11 of Table 1.)

8. The "theoretical" power for a a corrected 1-way analysis of variance on a predictor sort version of the data set:
$$\text{Prob}\left(\text{NCF}_{J-1,J(I-1)}(\gamma) > F^{-1}_{J-1,J(I-1)}(1-\alpha)\right) \tag{3}$$
where NCF denotes a non-central F distribution function, $\gamma = \sum_{j=1}^{J} I(\mu_j - \bar{\mu}.)^2/(\sigma^2_Y(1-\rho^2))$ is the non-centrality parameter of the noncentral F, and F^{-1} denotes the inverse of a central F distribution function. (The calculated "theoretical" power is reported in column 12 of Table 1.)

9. The "theoretical" power for a 2-way anova on a predictor sort version of the data set:
$$\text{Prob}\left(\text{NCF}_{J-1,(J-1)(I-1)}(\gamma) > F^{-1}_{J-1,(J-1)(I-1)}(1-\alpha)\right) \tag{4}$$
(The same non-centrality parameter is used in both (3) and (4).) (The calculated "theoretical" power is reported in column 13 of Table 1.)

The results of these simulations for the $J = 2, 5$; $I = 3, 5, 10, 20, 40$; $\rho = 0.5, 0.7, 0.9$ cases are presented in Table 1. The results for the remaining cases can be found at `http://www1.fpl.fs.fed.us/ps15_table1.html`. Plots that present a portion of the results of these simulations ($J = 2, 5$; $I = 10, 20$; $\rho = 0.0, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 0.99$) are attached as Figures 1–32. In these plots, the "noncentrality parameter index" is the m in column 4 of the corresponding power table. See Appendix A for a discussion of this index. "th" is the "theoretical" power calculated by (3) and presented in column 12 of the power table. "ps, anocov" is the power of an analysis of covariance after a predictor sort allocation (column 11 of the power table). "ps, two-way" is the power of a blocked analysis of variance after a predictor sort allocation (column 9 of the power table). "no ps, 1-way" is the power of a 1-way analysis of variance after a standard (non predictor sort) random allocation of specimens (column 5 of the power table). "ps, 1-way, no rho" is the power of an uncorrected 1-way analysis of variance after a predictor sort allocation (column 6 of the power table).

An analysis of these tables and plots yields the following conclusions:

1. Large increases in statistical power and/or sample size reductions can be gained by performing a predictor sort allocation and analysis. These improvements become larger as the correlation, ρ, between the predictor and the response increases. Specifically, if n samples are needed to achieve a given power when predictor sort allocation is not used, approximately $(1-\rho^2)n$ samples are needed to achieve the same power when predictor sort allocation is used. Thus,

for example, a 0.7 correlation yields, roughly, a halving of necessary sample size. (To see this, consider the $\rho = 0.7$ entries in Table 1 and at the `ps15_table1.html` web address provided above. For m in column 4 of the tables equal to, for example, 7, 12, 16, and 21, and $I = 5, 10, 20$, compare the powers listed in columns 8, 9, 11, 12, and 13 with the corresponding powers in column 5 for $I = 10, 20, 40$.)

2. It is a statistical *blunder* to perform a predictor sort allocation and then follow the allocation with a standard non predictor sort analysis of variance. Such an approach can considerably *reduce* power. (To see this, compare the values in column 6 of Table 1 with the values in columns 9, 11, and 12, or compare the ps,1-way,no rho lines in Figures 1 through 32 with the ps,two-way, ps,anocov, and th lines.)

3. After a predictor sort allocation has been performed, either an analysis of covariance or a blocked analysis of variance should be performed. For $\rho \leq 0.8$ and $I \geq 10$, the blocked analysis of variance performs almost as well as the analysis of covariance. For higher ρ and/or smaller I, the analysis of covariance performs better. (To see this, compare columns 9 and 11 in Table 1.)

4. For $\rho \leq 0.8$ and $I \geq 10$, the power of a blocked anova can be well approximated by (4). For higher ρ, (4) overestimates the power available from a blocked anova (especially for lower I). (To see this, compare columns 9 and 13 in Table 1.)

5. It is well known (also see results (47), (49), and (52) in Appendices B and C) that the power of a 1-factor analysis of covariance for testing the hypothesis $\mu_1 = \ldots = \mu_J$ is given by

$$\text{Prob}(\text{NCF}_{J-1,IJ-(J+1)}(\gamma) > F^{-1}_{J-1,IJ-(J+1)}(1-\alpha))$$

where, in our case, $\sigma^2_{\text{anocov}} = \sigma^2_Y(1-\rho^2)$, γ is the noncentrality parameter of the noncentral F, and

$$\begin{aligned}
\sigma^2_{\text{anocov}} \times \gamma &= \sum_{j=1}^{J} I(\mu_j - \bar{\mu}_.)^2 - \left(\sum_{j=1}^{J} I(\mu_j - \bar{\mu}_.)(\bar{x}_{.j} - \bar{x}_{..})\right)^2 / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{ij} - \bar{x}_{..})^2 \\
&\geq \sum_{j=1}^{J} I(\mu_j - \bar{\mu}_.)^2 - \left(\sum_{j=1}^{J} I(\mu_j - \bar{\mu}_.)^2 \sum_{j=1}^{J} I(\bar{x}_{.j} - \bar{x}_{..})^2\right) / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{ij} - \bar{x}_{..})^2 \\
&= \sum_{j=1}^{J} I(\mu_j - \bar{\mu}_.)^2 \left(1 - \sum_{j=1}^{J} I(\bar{x}_{.j} - \bar{x}_{..})^2 / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{ij} - \bar{x}_{..})^2\right) \qquad (5)
\end{aligned}$$

(The inequality in (5) is due to the Cauchy-Schwarz theorem.)

The simulations established that for $I \geq 5$, this power is well approximated by

$$\text{Prob}(\text{NCF}_{J-1,J(I-1)}(\hat{\gamma}) > F^{-1}_{J-1,J(I-1)}(1-\alpha))$$

where $\hat{\gamma} = \sum_{J=1}^{J} I(\mu_j - \bar{\mu}_.)^2/(\sigma^2_Y(1-\rho^2))$ is an approximation to the non-centrality parameter of the noncentral F. (To see this, compare columns 11 and 12 of Table 1.)

A listing of the simulation program that produced the Table 1 power estimates can be obtained at `http://www1.fpl.fs.fed.us/ps15_powersim_code.html`. A web-based simulation program

that can be run on additional cases (including multi-factor cases)
can be found at `http://www1.fpl.fs.fed.us/pspower.html`.

It can be argued that in a predictor sort situation a statistician would undoubtedly perform a blocked analysis (or an analysis of covariance using the predictor/concomitant as the covariate). However, some authors of statistical texts for non-statisticians (see, for example, some of the texts listed in Section 1) appear to treat "matched subject" allocations as good experimental practice regardless of the subsequent analyses. (For example, one of the texts discussed matching, t tests, and ANOVAs, but not paired t-tests, blocked ANOVAS, or analyses of covariance.) Given that very poor power can result if a predictor sort allocation is analyzed via an unblocked ANOVA, authors of (at least) statistical texts for non-statisticians need to make this clear. This is especially true for fields in which concomitants might be highly correlated with responses.

3 Confidence Intervals on Means

Verrill (1999) established the following theorem.

Theorem 2

Assume that the predictor variable and the variable of interest, Y, have a joint bivariate normal distribution with correlation ρ. Denote the variance of Y by σ_Y^2. Suppose that there are I blocks and F factors with $K_1, \ldots, K_F$ levels. Let the allocation of samples be as described in Section 1. (For a multiple factor case, enough adjacent experimental units would be chosen at a time to provide one additional observation for each cell.) Let $\bar{Y}_{\cdot j_1 \cdot \ldots \cdot}$ be the standard estimate of the mean response for the j_1th level of factor 1. Then

$$\sqrt{I \times K_2 \times \ldots \times K_F}\left(\bar{Y}_{\cdot j_1 \cdot \ldots \cdot} - \mathrm{E}\left(\bar{Y}_{\cdot j_1 \cdot \ldots \cdot}\right)\right) \xrightarrow{D} \mathrm{N}\left(0, \sigma_Y^2(1 - \rho^2 + \rho^2/K_1)\right)$$

as $I \to \infty$. The analogous results hold for factors $2, \ldots, F$.

Why can this result lead to problems?

If the predictor sort nature of an experiment is neglected, then the confidence interval that is constructed for the mean response associated with level j_1 of factor 1 is

$$\bar{y}_{\cdot j_1 \cdot \ldots \cdot} \pm t \times s/\sqrt{I \times K_2 \times \ldots \times K_F} \tag{6}$$

where t is the appropriate critical value, and s is the root mean residual sum of squares from the ANOVA. Verrill (1993) established (see the appendix of the 1993 paper, or Appendix G of the current paper) that in a predictor sort case, if the problem is treated as a $K_1 \times \ldots \times K_F$ ANOVA with I replicates per cell, the mean residual sum of squares, $s^2_{\text{unblocked}}$, satisfies

$$s^2_{\text{unblocked}} \xrightarrow{p} \sigma_Y^2 \tag{7}$$

as I increases to infinity. If the problem is treated as as one involving I blocks with 1 replicate per cell, the mean residual sum of squares, s^2_{blocked}, satisfies

$$s^2_{\text{blocked}} \xrightarrow{p} (1 - \rho^2)\sigma_Y^2 \tag{8}$$

where ρ is the correlation between the predictor used in the sort and Y.

But by Theorem 2 above, the appropriate large sample value for s in (6) is

$$\sigma_Y \sqrt{1 - \rho^2 + \rho^2/K_1}$$

rather than σ_Y or $\sigma_Y\sqrt{1-\rho^2}$. This discrepancy is the source of the coverage problems.

Let

$$R_{\rm ub}(\rho,J) \equiv 1/\left((1-\rho^2+\rho^2/J)\right)^{1/2}$$

and

$$R_{\rm b}(\rho,J) \equiv \left((1-\rho^2)/(1-\rho^2+\rho^2/J)\right)^{1/2}.$$

(Notice that we are switching from the "K" treatments notation of the 1999 paper to a "J" treatments notation here.)

In Figure 33 values of $R_{\rm ub}(\rho,J)$ are plotted. These R values approximate the factor by which confidence interval sizes are incorrectly inflated when a standard unblocked ANOVA is performed in a predictor sort case.

In Figure 34 values of $R_{\rm b}(\rho,J)$ are plotted. These values approximate the factor by which confidence interval sizes are incorrectly deflated when a standard blocked ANOVA is performed in a predictor sort case.

In Figure 35 values of

$$2\times\Phi\left(\Phi^{-1}(.975)\times R_{\rm ub}(\rho,J)\right)-1$$

are plotted where Φ denotes the cumulative distribution function of a N(0,1). These values approximate the actual confidence levels that are associated with nominal 95% confidence intervals in the unblocked case.

Finally, in Figure 36 values of

$$2\times\Phi\left(\Phi^{-1}(.975)\times R_{\rm b}(\rho,J)\right)-1$$

are plotted. These values approximate the actual confidence levels that are associated with nominal 95% confidence intervals in the blocked case.

From these plots it is clear that, given a predictor sort design, for higher ρ values, the confidence interval lengths and coverages produced by standard ANOVA analyses are unacceptable.

Verrill (1999) suggested two possible fixes to incorrect confidence intervals on the μ_j's (1-factor treatment means) in the predictor sort case. First, he noted that the s in (6) could be "corrected" by multiplying it by an estimate of $\sqrt{1-\rho^2+\rho^2/J}$ in the unblocked case or by an estimate of $\sqrt{1-\rho^2+\rho^2/J}/\sqrt{1-\rho^2}$ in the blocked case. He then performed simulations that indicated the number of replications that would be needed to ensure that these asymptotically correct adjustments would yield good confidence interval coverages. These numbers were reported in his tables 1 and 2. These tables indicated that for larger ρ's, fairly large sample sizes would be needed to ensure good μ_j confidence interval coverages. This problem appears to be driven by the sensitivity of the corrections to $\hat{\rho}$.

We have since realized that it is possible to avoid this problem by making use of results (7) and (8). Together, they imply that

$$s^2_{\rm unblocked}-s^2_{\rm blocked}\xrightarrow{p}\rho^2\sigma^2_Y$$

and

$$s^2_{\rm blocked}+\left(s^2_{\rm unblocked}-s^2_{\rm blocked}\right)/J\xrightarrow{p}(1-\rho^2+\rho^2/J)\sigma^2_Y$$

so, in the 1-factor case, we can take the corrected "anova z" confidence interval on μ_j to be

$$\bar{y}_{\cdot j}\pm z\left(\sqrt{s^2_{\rm blocked}+\left(s^2_{\rm unblocked}-s^2_{\rm blocked}\right)/J}\right)/\sqrt{I} \qquad (9)$$

where $z = \Phi^{-1}(1-\alpha/2)$ for a $1-\alpha$ confidence level, and the corrected "anova t" confidence interval on μ_j to be

$$\bar{y}_{\cdot j} \pm t \left(\sqrt{s^2_{\text{blocked}} + \left(s^2_{\text{unblocked}} - s^2_{\text{blocked}}\right)/J} \right) / \sqrt{I} \tag{10}$$

where $t = T^{-1}_{J(I-1)}(1-\alpha/2)$ for a $1-\alpha$ confidence level, and $T_{J(I-1)}$ denotes the cumulative distribution function of a t distribution with $J(I-1)$ degrees of freedom (this is an ad hoc choice for degrees of freedom).

The second solution that was blithely and incorrectly proposed by Verrill (1999) was an analysis of covariance. Recall (equation (2)) that in a 1-factor predictor sort case, we have

$$Y_{ij} = \mu_j + \sigma_Y \left(\rho \left(X_{k(i,j),n} - \mu_X \right) / \sigma_X + \sqrt{1-\rho^2} P_{ij} \right)$$

or

$$\begin{aligned} Y_{ij} &= \mu_j - \rho\, \sigma_Y\, \mu_X/\sigma_X + \rho\, \sigma_Y\, X_{k(i,j),n}/\sigma_X + \sqrt{1-\rho^2}\, \sigma_Y\, P_{ij} \\ &= a_j + b\, X_{k(i,j),n} + \sqrt{1-\rho^2}\, \sigma_Y\, P_{ij} \end{aligned} \tag{11}$$

where

$$a_j = \mu_j - \rho\, \sigma_Y\, \mu_X/\sigma_X$$

and

$$b = \rho\, \sigma_Y/\sigma_X$$

Now

$$a_j + b\, \bar{x}_{\cdot\cdot} = \mu_j - \rho(\sigma_Y/\sigma_X)\mu_X + \rho(\sigma_Y/\sigma_X)\bar{x}_{\cdot\cdot}$$

is an approximation to μ_j and, *given* the x's and model (11), it is well known that $\hat{a}_j + \hat{b}\bar{x}_{\cdot\cdot}$ has variance

$$\left(1/I + (\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})^2 / \left(\sum_{k=1}^{J} \sum_{i=1}^{I} (x_{ik} - \bar{x}_{\cdot k})^2 \right) \right) (1-\rho^2)\sigma_Y^2 \tag{12}$$

For large I this is of the order

$$(1-\rho^2)\sigma_Y^2/I$$

This value is an underestimate of the variance of $\hat{a}_j + \hat{b}\bar{x}_{\cdot\cdot}$. Heuristically, this is essentially due to the fact that we are treating $\bar{x}_{\cdot\cdot}$ as a constant when, instead,

$$\begin{aligned} \text{Var}(a_j + b \times \bar{x}_{\cdot\cdot}) &= b^2\sigma_X^2/(IJ) \\ &= \rho^2\sigma_Y^2/\sigma_X^2 \times \sigma_X^2/(IJ) \\ &= \sigma_Y^2\, \rho^2/(IJ) \end{aligned}$$

In Appendix C we show that

$$\hat{a}_j + \hat{b}\bar{x}_{\cdot\cdot} = \bar{y}_{\cdot j} - \hat{b}(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}) = \bar{y}_{\cdot j} - \hat{\rho}(\hat{\sigma}_Y/\hat{\sigma}_X)(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}) = \hat{\mu}_j \tag{13}$$

where $\hat{a}_j$ and $\hat{b}$ are the standard analysis of covariance estimators, and $\hat{\rho}$, $\hat{\sigma}_Y$, $\hat{\sigma}_X$, and $\hat{\mu}_j$ are maximum likelihood estimators obtained in appendix A.2 of Verrill *et al.* (2004). In Appendix D we show that under maximum likelihood regularity conditions (which we do not verify at this point)

$$\sqrt{I}(\hat{\mu}_j - \mu_j) \xrightarrow{\text{D}} \text{N}(0, \sigma_Y^2(1-\rho^2+\rho^2/J)) \tag{14}$$

From results (13) and (14) we have

$$\sqrt{I}(\bar{y}_{\cdot j} - \hat{b}(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}) - \mu_j) \xrightarrow{\text{D}} \text{N}(0, \sigma_Y^2(1 - \rho^2 + \rho^2/J)) \tag{15}$$

Thus, in the 1-factor case, we can take the corrected "anocov z" confidence interval on μ_j to be

$$\bar{y}_{\cdot j} - \hat{b}(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}) \pm z \left(\sqrt{s^2_{\text{blocked}} + \left(s^2_{\text{unblocked}} - s^2_{\text{blocked}}\right)/J} \right) / \sqrt{I} \tag{16}$$

where $z = \Phi^{-1}(1 - \alpha/2)$ for a $1 - \alpha$ confidence level, and the corrected "anocov t" confidence interval on μ_j to be

$$\bar{y}_{\cdot j} - \hat{b}(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}) \pm t \left(\sqrt{s^2_{\text{blocked}} + \left(s^2_{\text{unblocked}} - s^2_{\text{blocked}}\right)/J} \right) / \sqrt{I} \tag{17}$$

where $t = T^{-1}_{JI-(J+1)}(1 - \alpha/2)$ for a $1 - \alpha$ confidence level (again, the degrees of freedom are ad hoc).

Based on result (14), we take the maximum likelihood confidence interval to be

$$\hat{\mu}_j \pm z\,\hat{\sigma}_Y \sqrt{1 - \hat{\rho}^2 + \hat{\rho}^2/J}/\sqrt{I} \tag{18}$$

We have performed simulations on 1-factor predictor sort anovas and anocovs to evaluate the resulting confidence interval coverages. For all combinations of X,Y correlations 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, and 0.99; number of factor levels, J, equal to 2, 3, 5, 7, 9, 11, and 20; sample sizes, I, equal to 3, 5, 10, 20, 40, and 80; and confidence levels 90% and 95%, we performed 10,000 trials. For each of the resulting data sets we calculated standard unblocked and blocked anova confidence intervals on μ_1, corrected z and t anova confidence intervals on μ_1, a "standard" anocov confidence interval on μ_1, corrected z and t anocov confidence intervals on μ_1, and the maximum likelihood confidence interval on μ_1.

In accord with the reasoning associated with (12), we calculated the "standard" anocov confidence interval as

$$\bar{y}_{\cdot j} - \hat{b}(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}) \pm t s_{\text{anocov}} \sqrt{1/I + (\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})^2 / \left(\sum_{k=1}^{J} \sum_{i=1}^{I} (x_{ik} - \bar{x}_{\cdot k})^2 \right)} \tag{19}$$

The results of the 95% confidence interval simulations are presented in Table 2. The results of both the 90% and 95% confidence interval simulations
are available at `http://www1.fpl.fs.fed.us/ps15_table2.html`. In these tables, columns 4 and 5 report the coverages of standard 1-way and 2-way anovas after a predictor sort allocation. Columns 6 and 7 report the coverages of the confidence intervals given in (9) and (10). Column 8 reports the coverages of the standard (and thus incorrect) anocov confidence interval given in (19). Columns 9 and 10 report the coverages of the confidence intervals given in (16) and (17). Finally, column 11 reports the coverage of the maximum likelihood confidence interval given by (18).

For the 95% confidence interval simulations, we fit the model

$$\text{coverage} - .95 = c_1/I^{1/2} + c_2/I + c_3/I^{3/2}$$

to the tabled coverages, and then used the resulting fits to estimate the I's at which the actual coverages would first fall between .94 and .96. As an illustration, the data and fits for the $J = 3$, $\rho = 0.8$ case are plotted in Figures 37 (anova) and 38 (anocov). The estimated needed I's are provided in Table 3. For blocked anovas, these I's are much improved over the comparable values reported in Verrill's (1999) table 2. For unblocked anovas and $J \geq 5$, these I's are much improved over the comparable values reported in Verrill's (1999) table 1.

It is clear from the 95% and 90% confidence interval simulation tables that

1. The confidence interval coverage simulation results are in accord with the large sample results expressed in Figures 35 and 36. That is, for higher ρs, an uncorrected 1-factor/unblocked ANOVA will lead to confidence interval coverages that are larger than the nominal coverages (see column 4 of Table 2) and a 1-factor/blocked ANOVA will lead to coverages that are lower than nominal coverages (see column 5 of Table 2). Also, as expected from the discussion in connection with result (12), uncorrected anocov analyses will lead to actual coverages smaller than nominal coverages (see column 8 of Table 2).

2. Corrected anova's and anocov's yield good coverages for reasonable sample sizes. (See columns 6, 7, 9, and 10 of Table 2.)

3. For $\rho \leq 0.80$, good coverages are obtained most quickly/for the smallest sample sizes by taking a "uses t" approach. That is, we use the appropriate t critical value rather than the appropriate z critical value. For $\rho \geq 0.90$, corrected anova's yield correct coverages most quickly if a "uses t" aproach is taken for $J = 2$ and a "uses z" approach is taken otherwise. For $\rho \geq 0.90$, corrected anocov's yield correct coverages most quickly if a "uses t" aproach is taken for $J = 2, 3$ and a "uses z" approach is taken otherwise. (See columns 6, 7, 9, and 10 of Table 2.)

4. The maximum likelihood true coverage is slow to converge to the nominal coverage. (See column 11 of Table 2.)

A listing of the simulation program that produced the Table 2 coverage estimates can be obtained at `http://www1.fpl.fs.fed.us/ps15_confsim_code.html`. A web-based simulation program that can be run on additional cases (including multi-factor cases)
can be found at `http://www1.fpl.fs.fed.us/psconf.html`.

We note that under the conditions of Theorem 2, for factor 1, the multi-factor versions of confidence intervals (9), (10), (16), and (17) have the forms

$$\bar{y}_{\cdot j_1 \cdot \ldots \cdot} \pm z \times \left(\sqrt{s^2_{\text{blocked}} + \left(s^2_{\text{unblocked}} - s^2_{\text{blocked}}\right)/K_1} \right) / \sqrt{I \times K_2 \times \ldots \times K_F} \tag{20}$$

$$\bar{y}_{\cdot j_1 \cdot \ldots \cdot} \pm t \times \left(\sqrt{s^2_{\text{blocked}} + \left(s^2_{\text{unblocked}} - s^2_{\text{blocked}}\right)/K_1} \right) / \sqrt{I \times K_2 \times \ldots \times K_F} \tag{21}$$

$$\bar{y}_{\cdot j_1 \cdot \ldots \cdot} - \hat{b}(\bar{x}_{\cdot j_1 \cdot \ldots \cdot} - \bar{x}_{\cdot \ldots \cdot}) \pm z \times \left(\sqrt{s^2_{\text{blocked}} + \left(s^2_{\text{unblocked}} - s^2_{\text{blocked}}\right)/K_1} \right) / \sqrt{I \times K_2 \times \ldots \times K_F} \tag{22}$$

$$\bar{y}_{\cdot j_1 \cdot \ldots \cdot} - \hat{b}(\bar{x}_{\cdot j_1 \cdot \ldots \cdot} - \bar{x}_{\cdot \ldots \cdot}) \pm t \times \left(\sqrt{s^2_{\text{blocked}} + \left(s^2_{\text{unblocked}} - s^2_{\text{blocked}}\right)/K_1} \right) / \sqrt{I \times K_2 \times \ldots \times K_F} \tag{23}$$

where $z = \Phi^{-1}(1-\alpha/2)$ for a $1-\alpha$ confidence level, and $t = T^{-1}_{K_1 \ldots K_F I - (K_1 + K_2 - 1 + \ldots + K_F - 1 + 1)}(1-\alpha/2)$ for a $1 - \alpha$ confidence level (again, the degrees of freedom are ad hoc). The analogous results hold for factors $2, \ldots, F$.

4 Confidence Bounds on Quantiles

Verrill, Herian, and Green (2004) addressed a more specialized problem associated with predictor sort allocations.

Designers working with lumber must try to ensure that the strengths of wood structural members exceed the loads to which the members will be subjected. One approach to this problem is to design so that expected loads do not exceed "allowable strength properties" associated with particular species and grades of lumber (ASTM D1990). An allowable property is commonly obtained experimentally by taking a sample from the lumber population in question, obtaining a lower one-sided confidence bound on the fifth percentile of the strength distribution of the population, and then dividing by safety and duration-of-load factors.

If a normal strength distribution is assumed, engineers working with solid-sawn lumber (see, for example, ASTM D2915) can obtain a parametric one-sided lower confidence bound on the fifth percentile via the formula:

$$\bar{Y}_I - k_{I,\alpha,\beta} S_I \tag{24}$$

where we want to cover the α quantile with confidence $\beta \times 100\%$ and we have I replicates. Here $\bar{Y}_I$ denotes the average of I strength measurements, and S_I denotes the sample standard deviation of the measurements. Guttman (1970, Table 4.6) provides k values for $I = 2(1)100$, $(10)300$, $(25)500$, $(50)700$, $(100)1000$, $\alpha = 0.01$, 0.05, 0.10, and 0.25, and $\beta = 0.75$, 0.90, 0.95, and 0.99. He credits Owen (1963) for these tables. The approach is based on the noncentral t distribution and is exact if the experimental design involves standard random sampling rather than predictor sort sampling.

Scientists in other areas (e.g., composite materials, groundwater monitoring, and soil remediation) also make use of formula (24) to obtain confidence bounds on quantiles. See, for example, MIL-HDBK-17-1 (2003), Gibbons (1994), and Michigan DEQ (1994).

For formula (24) to be valid, the sample of lumber (or composite material, water, soil, ...) must be a standard random sample. However, as noted in the Introduction, wood strength researchers sometimes replace experimental unit allocation via random sampling with allocation via sorts based on non-destructive measurements of strength predictors such as modulus of elasticity and specific gravity. That is, they perform predictor sort experiments.

In Verrill *et al.* (2004), the authors examined the effect of predictor sort sampling on one-sided confidence bounds for normal quantiles. They found that standard noncentral t theory that ignores the predictor sort nature of the sampling leads to $\bar{Y} - kS$ bounds that are too low and thus statistically conservative (actual confidence levels are greater than nominal levels). On the other hand, maximum likelihood methods yield bounds that are too high and thus statistically non-conservative even for fairly large sample sizes.

The authors used tools developed in Verrill (1993, 1999) to establish an asymptotic result that yields the appropriate corrections for the standard noncentral t approach.

Theorem 3

Assume that the predictor variable and the variable of interest, Y, have a joint bivariate normal distribution with correlation ρ. Denote the variance of Y by σ_Y^2. Suppose that we have a 1-factor ANOVA design with I blocks and J treatments, and the allocation of samples is done via a predictor sort as described in Section 1.

Let Y_{ij} be given by (2). Define

$$\bar{Y}_{\cdot j} \equiv \sum_{i=1}^{I} Y_{ij}/I$$

and

$$S^2 = \sum_{j=1}^{J}\sum_{i=1}^{I}(Y_{ij} - \bar{Y}_{\cdot j})^2/(IJ-1)$$

and let $\hat{\rho}$ be any consistent estimator of ρ. Then,

$$\text{Prob}\left(\bar{Y}_{\cdot j} - \hat{k}_{I,\alpha,\beta}S \leq \mu_j + \Phi^{-1}(\alpha)\,\sigma_Y\right) \to \beta$$

as $I \to \infty$, where μ_j is the mean response for the jth treatment, Φ^{-1} denotes the inverse of a standard normal cumulative distribution function,

$$\hat{k}_{I,\alpha,\beta} \equiv \sqrt{(1-\hat{\rho}^2+\hat{\rho}^2/J)/I}\,\text{NCT}^{-1}_{IJ-1,\gamma_I(\hat{\rho})}(\beta),$$

$\text{NCT}^{-1}_{IJ-1,\gamma_I(\hat{\rho})}$ denotes the inverse of a noncentral t distribution with $IJ-1$ degrees of freedom and noncentrality parameter $\gamma_I(\hat{\rho})$, and

$$\gamma_I(\hat{\rho}) = -\Phi^{-1}(\alpha)\sqrt{I}\left(1-\hat{\rho}^2+\hat{\rho}^2/J\right)^{-1/2}$$

Proof
The proof is provided in Appendix B of Verrill *et al.* (2004).

In tables 1 to 24 of Verrill *et al.* (2004), the authors detail the coverages of four kinds of confidence interval on a quantile for a variety of combinations of ρ, α, β, number of treatments (J), and number of replicates (I). The rows of the tables are based on separate 4,000 trial simulations. The four approaches considered were the incorrect standard approach given in (24), which ignores the dependencies induced by sorting on the predictor; two versions of the (correct) predictor sort $\bar{Y} - kS$ asymptotic approach described in Theorem 3; and a maximum likelihood approach. The two versions of the predictor sort approach differ in the estimate used for the correlation between the predictor and the response. Version 1 uses the consistent estimate

$$\hat{\rho} \equiv \sum_{j=1}^{J}\sum_{i=1}^{I}(X_{ij}-\bar{X}_{\cdot\cdot})(Y_{ij}-\bar{Y}_{\cdot j})/\sqrt{\sum_{j=1}^{J}\sum_{i=1}^{I}(X_{ij}-\bar{X}_{\cdot\cdot})^2\sum_{j=1}^{J}\sum_{i=1}^{I}(Y_{ij}-\bar{Y}_{\cdot j})^2}$$

Version 2 uses the maximum likelihood estimate of ρ. It is clear from the tables that the incorrect approach is overly conservative, that the problem becomes more severe as the correlation between the predictor and response variables increases, and that the problem does not vanish as sample sizes increase. It is also clear from the tables that version 2 of the predictor sort approach dominates the maximum likelihood approach in the sense that the actual coverage always approaches the nominal coverage more rapidly for the version 2 predictor sort approach than for the maximum likelihood approach. For smaller J, the version 1 predictor sort approach performs better than the maximum likelihood approach and the version 2 approach (see Figure 39 for an example). However, for large J and small I, the version 1 approach does not perform as well.

For smaller I the asymptotic approaches are non-conservative. The authors have developed a program that can be run interactively to yield the appropriate k values for small sample sizes. This program will be described in an upcoming Forest Products Laboratory technical report.

4.1 Reductions in the sample sizes needed to obtain confidence bounds on quantiles given predictor sort sampling

In the course of the development of their asymptotic theory Verrill *et al.* (2004) found that the correct k in the appropriate version of $\bar{Y} - kS$ is given by

$$k \approx -\Phi^{-1}(\alpha) + \Phi^{-1}(\beta)I^{-1/2}\sqrt{(\Phi^{-1}(\alpha))^2/(2J) + 1 - \rho^2 + \rho^2/J}$$

where Φ denotes the N(0,1) distribution function. Thus given higher ρ values, we can have smaller I values, and still have the same k. In fact if we set

$$I^{-1/2}\sqrt{(\Phi^{-1}(\alpha))^2/(2J) + 1 - \rho^2 + \rho^2/J}$$

equal to a constant we obtain

$$I \propto (\Phi^{-1}(\alpha))^2/(2J) + 1 - \rho^2 + \rho^2/J$$

Thus the approximate permissible sample size reduction factor obtained by using a predictor sort with a correlation of ρ between the predictor and the response is (here the denominator is the numerator with ρ set equal to 0)

$$\left((\Phi^{-1}(\alpha))^2/(2J) + 1 - \rho^2 + \rho^2/J\right) / \left((\Phi^{-1}(\alpha))^2/(2J) + 1\right)$$

For a confidence bound on the 0.05 quantile, sample size reduction factors as functions of ρ and J are plotted in Figure 40. It is clear from the figure that practically significant sample size reductions (e.g., 30%) are attainable for correlations as low as 0.70.

5 Scheffé and Tukey multiple comparison procedures after a predictor sort allocation

As one would expect given the hypothesis test results established in Verrill (1993), for large sample sizes, suitably altered versions of the Scheffé and Tukey multiple comparison procedures are valid after a predictor sort allocation. In this section, we describe the alterations and establish the needed asymptotic results.

We first introduce the notation that we will use in this section. Assume that we have F factors, K_j levels for the jth factor, and I blocks (formed by specimens with adjacent [randomized within a block] order statistics of the predictor). Let $n \equiv IK_1 \ldots K_F$, and $\{X_i, i = 1, \ldots, n\}$, $\{Z_i, i = 1, \ldots, n\}$ be i.i.d. N(0,1) random variables. Define $W_i \equiv \sigma_Y \left(\rho X_i + \sqrt{1-\rho^2}\, Z_i\right)$.

We model predictor sort allocation by ordering the X's (the predictors), and randomly dividing the W's that correspond to $X_{(i-1)K_1 \ldots K_F+1,n}, \ldots, X_{iK_1 \ldots K_F,n}$ among the $K_1 \times K_2 \times \ldots \times K_F$ treatments. (Here, $X_{l,n}$ is the lth order statistic among the X's.)

Let $Y_{ij_1 \ldots J_F}$ denote $\mu_{j_1 \cdot \ldots \cdot} + \ldots + \mu_{\cdot \ldots \cdot j_F}$ plus the ith W that is assigned to treatment $j_1 \ldots j_F$, where, for example, $\mu_{j_1 \cdot \ldots \cdot}$ denotes the effect associated with the j_1th level of factor 1, and $\mu_{\cdot \ldots \cdot j_F}$ denotes the effect associated with the j_Fth level of factor F. Then

$$Y_{ij_1 \ldots j_F} = \mu_{j_1 \cdot \ldots \cdot} + \ldots + \mu_{\cdot \ldots \cdot j_F} + \sigma_Y \left(\rho X_{k(ij_1 \ldots \mathrm{J}_F),n} + \sqrt{1-\rho^2}\, P_{ij_1 \ldots j_F}\right) \tag{25}$$

where $k(ij_1 \ldots j_F) \in \{(i-1)K_1 \ldots K_F + 1, \ldots, iK_1 \ldots K_F\}$, and the $P_{ij_1 \ldots j_F}$ are i.i.d. N(0,1) and are independent of the X's.

(Note that there is some ugly notation here. The "$ij_1 \ldots j_F$" in $k(ij_1 \ldots j_F)$ is not a product and $k(ij_1 \ldots j_F)$ should really be written as $k(i, j_1, \ldots, j_F)$, but for simplicity, we omit the commas. On the other hand, "$iK_1 \ldots K_F$" actually is a product.)

5.1 Scheffé's multiple comparison test/procedure

The result is basically simple, but we need to provide the notation and describe the approach to express it. This approach is, of course, due to Scheffé.

First we essentially restate Scheffé's result in the non predictor sort case and then we identify those aspects of the proof that need to be altered in the predictor sort case.

Assume that

$$\mathbf{y} \sim \mathrm{N}(\mathbf{X}\boldsymbol{\beta}, \mathbf{I}\sigma^2) \tag{26}$$

(We note that the X in equation (25) is distinct from the $\mathbf{X}$ in formula (26).) Let $\mathbf{c}_1, \ldots, \mathbf{c}_q$ correspond to q linearly independent estimable functions. That is, the $\mathbf{c}$'s are linearly independent and there exist $\mathbf{a}_1, \ldots, \mathbf{a}_q$ that lie in the linear span of the columns of the design matrix $\mathbf{X}$ such that

$$\mathbf{a}_j^{\mathrm{T}}\mathbf{X} = \mathbf{c}_j^{\mathrm{T}}$$

for $j \in 1, \ldots, q$ so

$$E(\mathbf{a}_j^{\mathrm{T}}\mathbf{y}) = \mathbf{a}_j^{\mathrm{T}}\mathbf{X}\boldsymbol{\beta} = \mathbf{c}_j^{\mathrm{T}}\boldsymbol{\beta}$$

for $j \in 1, \ldots, q$.

We have

$$\mathbf{v} \equiv \begin{pmatrix} \mathbf{a}_1^{\mathrm{T}}\mathbf{y} - \mathbf{c}_1^{\mathrm{T}}\boldsymbol{\beta} \\ \vdots \\ \mathbf{a}_q^{\mathrm{T}}\mathbf{y} - \mathbf{c}_q^{\mathrm{T}}\boldsymbol{\beta} \end{pmatrix} \sim N\left(\mathbf{0}, \mathbf{A}\sigma^2\right)$$

or

$$\mathbf{v} \equiv \begin{pmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \vdots \\ \mathbf{a}_q^{\mathrm{T}} \end{pmatrix} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \sim N\left(\mathbf{0}, \mathbf{A}\sigma^2\right)$$

where

$$\mathbf{A} \equiv \begin{pmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \vdots \\ \mathbf{a}_q^{\mathrm{T}} \end{pmatrix} (\mathbf{a}_1 \ldots \mathbf{a}_q)$$

Thus,

$$\mathbf{A}^{-1/2}\mathbf{v} \sim N\left(\mathbf{0}, \mathbf{I}_{q\times q}\sigma^2\right)$$

so

$$\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{v}/\sigma^2 \sim \chi_q^2 \tag{27}$$

Now, by the Cauchy-Schwarz theorem, *for all* $\mathbf{l} \in \mathbb{R}^q$,

$$\left(\mathbf{l}^{\mathrm{T}}\mathbf{v}\right)^2 = \left(\mathbf{l}^{\mathrm{T}}\mathbf{A}^{1/2}\mathbf{A}^{-1/2}\mathbf{v}\right)^2 \leq \left(\mathbf{l}^{\mathrm{T}}\mathbf{A}\mathbf{l}\right)\left(\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{v}\right) \tag{28}$$

so by result (27), *simultaneously for all* $\mathbf{l} \in \mathbb{R}^q$ (note that the right hand side of (29) does not depend on $\mathbf{l}$),

$$\left(\mathbf{l}^{\mathrm{T}}\mathbf{v}\right)^2 / \left(qs^2\mathbf{l}^{\mathrm{T}}\mathbf{A}\mathbf{l}\right) \leq \left(\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{v}/q\right)/s^2 \sim F_{q,n-r} \tag{29}$$

where n is the total number of observations, r is the rank of $\mathbf{X}$, and

$$s^2 = \left(\mathbf{y}^{\mathrm{T}}\mathbf{y} - \left((\mathbf{u}_1^{\mathrm{T}}\mathbf{y})^2 + \ldots + (\mathbf{u}_r^{\mathrm{T}}\mathbf{y})^2\right)\right)/(n-r)$$

is the mean residual sum of squares ($\mathbf{u}_1, \ldots, \mathbf{u}_r$ is an orthonormal basis of the space spanned by the columns of $\mathbf{X}$).

So how does result (29) change in a predictor sort situation?

For the first factor, we will be testing the null hypothesis that $\mu_{1\cdot\ldots\cdot} = \mu_{2\cdot\ldots\cdot} = \ldots = \mu_{K_1\cdot\ldots\cdot}$ or that $\mathbf{c}^{\mathrm{T}}\boldsymbol{\mu} = 0$ provided that $\mathbf{c}^{\mathrm{T}}\mathbf{1} = 0$ (the "contrasts" are all 0). (For ease of presentation, in what follows, we will focus on the first factor, but similar results hold for the other $F-1$ factors.)

In this case, in our formulation of the problem, the first K_1 columns of $\mathbf{X}$ correspond to factor 1. The first column of $\mathbf{X}$ contains $K_2 \ldots K_F I$ ones followed by $(K_1-1)K_2 \ldots K_F I$ zeros, ..., the K_1th column of $\mathbf{X}$ contains $(K_1-1)K_2 \ldots K_F I$ zeros followed by $K_2 \ldots K_F I$ ones.

The next K_2 columns of $\mathbf{X}$ correspond to factor 2. The first $K_3K_4 \ldots K_F I$ rows of column K_1+1 of $\mathbf{X}$ contain ones. The next $(K_2-1)K_3K_4 \ldots K_F I$ rows of column K_1+1 of $\mathbf{X}$ contain zeros. This pattern is repeated K_1-1 more times to complete column K_1+1. The one's are shifted down systematically in the following columns until in column K_1+K_2, the first $(K_2-1)K_3K_4 \ldots K_F I$ rows of column K_1+K_2 of $\mathbf{X}$ contain zeros, the next $K_3K_4 \ldots K_F$ rows of column K_1+K_2 of $\mathbf{X}$ contain ones, and this pattern is repeated K_1-1 more times to complete column K_1+K_2.

This process is continued through the final K_F columns of $\mathbf{X}$ which correspond to the levels of factor F (or through the final I columns of $\mathbf{X}$ if we perform a blocked ANOVA).

For the specific multiple comparison problem that we are considering (we are interested in being able to test all of the contrasts of the first factor means), $\mathbf{c}_1$ contains a 1 in its first element, a -1 in its second element, and zeros in its remaining elements, ..., $\mathbf{c}_{K_1-1}$ contains a 1 in its first element, zeros in its next K_1-2 elements, a -1 in its K_1th element, and zeros in its remaining elements. The $\mathbf{a}_j$ corresponding to $\mathbf{c}_j$ has $1/(K_2 \ldots K_F I)$ in its first $K_2 \ldots K_F I$ elments, $-1/(K_2 \ldots K_F I)$ in elements $jK_2 \ldots K_F I+1, \ldots, (j+1)K_2 \ldots K_F I$, and zeros elsewhere. If $\mathbf{u}_1, \ldots, \mathbf{u}_{K_1-1}$ is an orthonormal basis of the linear span of $\mathbf{a}_1, \ldots, \mathbf{a}_{K_1-1}$, then

$$\begin{aligned}
\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{v} &= (\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^{\mathrm{T}} (\mathbf{a}_1 \ldots \mathbf{a}_{K_1-1})\, \mathbf{A}^{-1} \begin{pmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \vdots \\ \mathbf{a}_{K_1-1}^{\mathrm{T}} \end{pmatrix} (\mathbf{y}-\mathbf{X}\boldsymbol{\beta}) \\
&= (\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^{\mathrm{T}} (\mathbf{u}_1 \ldots \mathbf{u}_{K_1-1}) \begin{pmatrix} \mathbf{u}_1^{\mathrm{T}} \\ \vdots \\ \mathbf{u}_{K_1-1}^{\mathrm{T}} \end{pmatrix} (\mathbf{y}-\mathbf{X}\boldsymbol{\beta}) \qquad (30)
\end{aligned}$$

Define $n \equiv K_1 \ldots K_F I$. It is clear that $\mathbf{u}_1, \ldots, \mathbf{u}_{K_1-1}, \mathbf{1}/\sqrt{n}$ have the same linear span as $\mathbf{w}_1, \ldots, \mathbf{w}_{K_1}$ where $\mathbf{w}_j$ contains $1/\sqrt{K_2 \ldots K_F I}$ in rows $(j-1)K_2 \ldots K_F I+1, \ldots, jK_2 \ldots K_F I$ and zeros elsewhere. Thus,

$$\begin{aligned}
\mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{v} &= \sum_{j_1=1}^{K_1} \left(\mathbf{w}_{j_1}^{\mathrm{T}}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right)^2 - \left(\mathbf{1}^{\mathrm{T}}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})/\sqrt{n}\,\right)^2 \\
&= \sum_{j_1=1}^{K_1} \left(\mathbf{w}_{j_1}^{\mathrm{T}}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right)^2 - n\,(\bar{y}_{\cdot\ldots\cdot} - (\mu_{-\cdot\ldots\cdot} + \ldots + \mu_{\cdot\ldots\cdot-}))^2 \\
&= \sum_{j_1=1}^{K_1} K_2 \ldots K_F I\,(\bar{y}_{\cdot j_1 \cdot\ldots\cdot} - (\mu_{j_1\cdot\ldots\cdot} + \mu_{\cdot-\cdot\ldots\cdot} + \ldots + \mu_{\cdot\ldots\cdot-}))^2 \\
&- n\,(\bar{y}_{\cdot\ldots\cdot} - (\mu_{-\cdot\ldots\cdot} + \ldots + \mu_{\cdot\ldots\cdot-}))^2 \\
&= \sum_{j_1=1}^{K_1} K_2 \ldots K_F I\,(\bar{y}_{\cdot j_1 \cdot\ldots\cdot} - (\mu_{j_1\cdot\ldots\cdot} + \mu_{\cdot-\cdot\ldots\cdot} + \ldots + \mu_{\cdot\ldots\cdot-}) \\
&- (\bar{y}_{\cdot\ldots\cdot} - (\mu_{-\cdot\ldots\cdot} + \ldots + \mu_{\cdot\ldots\cdot-})))^2 \\
&= \sum_{j_1=1}^{K_1} K_2 \ldots K_F I \times \sigma_Y^2 \times \left(\rho(\bar{x}_{k(\cdot j_1\cdot\ldots\cdot),n} - \bar{x}_{k(\cdot\ldots\cdot),n}) + \sqrt{1-\rho^2}(\bar{p}_{\cdot j_1\cdot\ldots\cdot} - \bar{p}_{\cdot\ldots\cdot})\right)^2 \\
&\xrightarrow{\mathrm{D}} (1-\rho^2)\sigma_Y^2\chi^2_{K_1-1}
\end{aligned} \tag{31}$$

as $I \to \infty$ by the material in section 14.1. Here,

$$\mu_{-\cdot\ldots\cdot} \equiv \sum_{j_1=1}^{K_1} \mu_{j_1\cdot\ldots\cdot}/K_1$$

Similar definitions hold for $\mu_{\cdot-\cdot\ldots\cdot}, \ldots, \mu_{\cdot\ldots\cdot-}$.

Also, in sections 14.2 and 14.3, we show that in the "unblocked" predictor sort case,

$$s^2_{\mathrm{ub}} \xrightarrow{\mathrm{P}} \sigma_Y^2 \tag{32}$$

where

$$s^2_{\mathrm{ub}} = \mathrm{SS}_{\mathrm{den,unbl}}/(IK_1\ldots K_F - (K_1 + K_2 - 1 + \ldots + K_F - 1)) \tag{33}$$

and

$$\mathrm{SS}_{\mathrm{den,unbl}} = \sum_{i=1}^{I}\sum_{j_1=1}^{K_1}\ldots\sum_{j_F=1}^{K_F}(y_{ij_1\ldots j_F} - (\bar{y}_{\cdot\ldots\cdot} + (\bar{y}_{\cdot j_1\cdot\ldots\cdot} - \bar{y}_{\cdot\ldots\cdot}) + \ldots + (\bar{y}_{\cdot\ldots\cdot j_F} - \bar{y}_{\cdot\ldots\cdot})))^2$$

while, in the "blocked" predictor sort case,

$$s^2_{\mathrm{b}} \xrightarrow{\mathrm{P}} (1-\rho^2)\sigma_Y^2 \tag{34}$$

where

$$s^2_{\mathrm{b}} = \mathrm{SS}_{\mathrm{den,bl}}/(IK_1\ldots K_F - (I + K_1 - 1 + \ldots + K_F - 1)) \tag{35}$$

and

$$\mathrm{SS}_{\mathrm{den,bl}} = \sum_{i=1}^{I}\sum_{j_1=1}^{K_1}\ldots\sum_{j_F=1}^{K_F}(y_{ij_1\ldots j_F} - (\bar{y}_{\cdot\ldots\cdot} + (\bar{y}_{i\cdot\ldots\cdot} - \bar{y}_{\cdot\ldots\cdot}) + \ldots + (\bar{y}_{\cdot\ldots\cdot j_F} - \bar{y}_{\cdot\ldots\cdot})))^2$$

From results (28), (31), (32), and (34), we know that in the factor 1, unblocked predictor sort case we have, *simultaneously for all* $\mathbf{l} \in \mathbb{R}^{K_1-1}$,

$$\left(\mathbf{l}^{\mathrm{T}}\mathbf{v}\right)^2 / \left(s_{\mathrm{ub}}^2\,\mathbf{l}^{\mathrm{T}}\mathbf{A}\mathbf{l}\right) \le \mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{v}/s_{\mathrm{ub}}^2 \overset{\mathrm{D}}{\to} \left(1-\rho^2\right)\chi^2_{K_1-1}$$

and in the blocked predictor sort case we have

$$\left(\mathbf{l}^{\mathrm{T}}\mathbf{v}\right)^2 / \left(s_{\mathrm{b}}^2\,\mathbf{l}^{\mathrm{T}}\mathbf{A}\mathbf{l}\right) \le \mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{v}/s_{\mathrm{b}}^2 \overset{\mathrm{D}}{\to} \chi^2_{K_1-1}$$

Analogous results hold for factors $2-F$. So, we are led to the following theorem.

Theorem 4

Assume that the predictor variable and the variable of interest have a joint bivariate normal distribution with correlation ρ. Let the allocation of samples be as described in Section 1. (For the multi-factor case, enough adjacent experimental units are chosen at a time to provide one additional observation for each cell.) Given the notation provided above in association with results (30), (31), (32), and (34), where the $\mathbf{A}$ and $\mathbf{v}$ are appropriate for testing the contrasts of the factor j means, in the unblocked predictor sort case we have, *simultaneously for all* $\mathbf{l} \in \mathbb{R}^{K_j-1}$,

$$\left(\mathbf{l}^{\mathrm{T}}\mathbf{v}\right)^2 / \left(s_{\mathrm{ub}}^2\,\mathbf{l}^{\mathrm{T}}\mathbf{A}\mathbf{l}\right) \le \mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{v}/s_{\mathrm{ub}}^2 \overset{\mathrm{D}}{\to} \left(1-\rho^2\right)\chi^2_{K_j-1} \tag{36}$$

and in the blocked predictor sort case we have, *simultaneously for all* $\mathbf{l} \in \mathbb{R}^{K_j-1}$,

$$\left(\mathbf{l}^{\mathrm{T}}\mathbf{v}\right)^2 / \left(s_{\mathrm{b}}^2\,\mathbf{l}^{\mathrm{T}}\mathbf{A}\mathbf{l}\right) \le \mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{v}/s_{\mathrm{b}}^2 \overset{\mathrm{D}}{\to} \chi^2_{K_j-1} \tag{37}$$

5.2 Simultaneous confidence intervals on contrasts based on Scheffé's multiple comparison procedure

From result (36), in the unblocked predictor sort case, for the factor j (where $\mathbf{A}$ and $\mathbf{v}$ will depend on j), given any $\epsilon > 0$ and $\alpha \in (0,1)$, we can find an $N_{\epsilon,\alpha}$ such that $I > N_{\epsilon,\alpha}$ implies that

$$\mathrm{Prob}\left(\left(\mathbf{l}^{\mathrm{T}}\mathbf{v}\right)^2 \le s_{\mathrm{ub}}^2\,(\mathbf{l}^{\mathrm{T}}\mathbf{A}\mathbf{l})(1-\rho^2)F^{-1}_{\chi^2_{K_j-1}}(1-\alpha)\ \textit{for all}\ \mathbf{l} \in \mathbb{R}^{K_j-1}\right) > 1-\alpha-\epsilon$$

where $F_{\chi^2_{K_j-1}}$ is the cdf of a $\chi^2_{K_j-1}$, or

$$\mathrm{Prob}\left(\left(\mathbf{l}^{\mathrm{T}}\begin{pmatrix}\mathbf{a}_1^{\mathrm{T}}\\ \vdots \\ \mathbf{a}_{K_j-1}^{\mathrm{T}}\end{pmatrix}\mathbf{y} - \mathbf{l}^{\mathrm{T}}\begin{pmatrix}\mathbf{c}_1^{\mathrm{T}}\boldsymbol{\beta}\\ \vdots \\ \mathbf{c}_{K_j-1}^{\mathrm{T}}\boldsymbol{\beta}\end{pmatrix}\right)^2 \le s_{\mathrm{ub}}^2(\mathbf{l}^{\mathrm{T}}\mathbf{A}\mathbf{l})(1-\rho^2)F^{-1}_{\chi^2_{K_j-1}}(1-\alpha)\ \textit{for all}\ \mathbf{l} \in \mathbb{R}^{K_j-1}\right) > 1-\alpha-\epsilon$$

That is, for I large enough, the intervals

$$\mathbf{l}^{\mathrm{T}}\begin{pmatrix}\mathbf{a}_1^{\mathrm{T}}\\ \vdots \\ \mathbf{a}_{K_j-1}^{\mathrm{T}}\end{pmatrix}\mathbf{y} \pm s_{\mathrm{ub}}\sqrt{\mathbf{l}^{\mathrm{T}}\mathbf{A}\mathbf{l}}\sqrt{1-\rho^2}\sqrt{F^{-1}_{\chi^2_{K_j-1}}(1-\alpha)}$$

are simultaneous (for all $\mathbf{l} \in \mathbb{R}^{K_j-1}$), approximate $1-\alpha$ confidence intervals on the contrasts

$$\mathbf{l}^{\mathrm{T}}\begin{pmatrix}\mathbf{c}_1^{\mathrm{T}}\boldsymbol{\beta}\\ \vdots \\ \mathbf{c}_{K_j-1}^{\mathrm{T}}\boldsymbol{\beta}\end{pmatrix} \tag{38}$$

Similarly, from result (37), in the blocked predictor sort case, for I large enough, the intervals

$$\mathbf{l}^{\mathrm{T}} \begin{pmatrix} \mathbf{a}_1^{\mathrm{T}} \\ \vdots \\ \mathbf{a}_{K_j-1}^{\mathrm{T}} \end{pmatrix} \mathbf{y} \pm s_{\mathrm{b}} \sqrt{\mathbf{l}^{\mathrm{T}}\mathbf{A}\mathbf{l}} \sqrt{F^{-1}_{\chi^2_{K_j-1}}(1-\alpha)}$$

are simultaneous (for all $\mathbf{l} \in \mathbb{R}^{K_j-1}$), approximate $1-\alpha$ confidence intervals on the contrasts in (38).

Note that for a contrast

$$\mathbf{s}^{\mathrm{T}}\boldsymbol{\beta} = (s_1 \ldots s_{K_j})^{\mathrm{T}}\boldsymbol{\beta}$$

$(s_1 + \ldots + s_{K_j} = 0)$ we can obtain the necessary $\mathbf{l}$ via

$$\left(\mathbf{c}_1 \ldots \mathbf{c}_{K_j-1}\right)\mathbf{l} = \mathbf{s}$$

or

$$\begin{pmatrix} 1 & & 1 \\ -1 & & 0 \\ 0 & & \vdots \\ \vdots & \cdots & 0 \\ 0 & & -1 \end{pmatrix} \mathbf{l} = \mathbf{s}$$

so

$$\mathbf{l} = \begin{pmatrix} -s_2 \\ -s_3 \\ \vdots \\ -s_{K_j} \end{pmatrix}$$

5.3 Tukey's multiple comparison test/procedure

Theorem 5

Assume that the predictor variable and the variable of interest, Y, have a joint bivariate normal distribution with correlation ρ. Denote the variance of Y by σ_Y^2. Suppose that there are I blocks and F factors with $K_1, \ldots, K_F$ levels. Let the allocation of samples be as described in Section 1. (For a multiple factor case, enough adjacent experimental units would be chosen at a time to provide one additional observation for each cell.) Let $\bar{Y}_{\cdot j_1 \cdot \ldots \cdot}$ be the standard estimate of the mean response for the j_1th level of factor 1. For comparisons of the factor 1 levels, let the numerator of the test statistic be given by

$$Q_I \equiv \max_{l_1, l_2 \in \{1,\ldots,K_1\}} \sqrt{IK_2 \ldots K_F}\,|\bar{Y}_{\cdot l_1 \cdot \ldots \cdot} - \bar{Y}_{\cdot l_2 \cdot \ldots \cdot}|$$

Let s^2_{ub} (given by equation (33)) denote the estimate of σ^2 in the unblocked case, and s^2_{b} (given by equation (35)) denote the estimate of σ^2 in the blocked case.

Let $F_{\mathrm{R(K_1)}}$ denote the distribution of the range of a sample of K_1 independent N(0,1)'s.

Then, under the null hypothesis that $\mu_{1 \cdot \ldots \cdot} = \ldots = \mu_{K_1 \cdot \ldots \cdot}$,

$$Q_I / \left(s_{\mathrm{ub}}\sqrt{1-\rho^2}\right) \xrightarrow{\mathrm{D}} F_{\mathrm{R(K_1)}} \tag{39}$$

and

$$Q_I / s_{\mathrm{b}} \xrightarrow{\mathrm{D}} F_{\mathrm{R(K_1)}} \tag{40}$$

Similar results hold for factors 2 through F.

Proof

The proof appears in Appendix F.

We have performed simulations on 1-factor predictor sort anovas to evaluate the resulting sizes of Tukey tests. For all combinations of X,Y correlations 0.0, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, and 0.99, number of treatments, J, equal to 3, 5, 7, 9, 11, and 20, and sample sizes, I, equal to 3, 5, 10, 20, and 40, we performed 100,000 trials. For each of the resulting data sets we calculated a standard unblocked Tukey test statistic on the equivalence of the J treatment means, an unblocked Tukey test statistic that has been corrected by an estimated $\sqrt{1-\rho^2}$ factor, an unblocked Tukey test statistic that has been corrected by the true $\sqrt{1-\rho^2}$ factor, and a blocked Tukey test statistic. In Table 4, we report the resulting actual test sizes when the nominal size is 0.05.

We can conclude from this table that

1. For the non-zero ρ's considered, a standard unblocked Tukey test (one that uses an s_{ub} denominator with no ρ correction) yields test sizes that can be much less than the nominal test size. (See column 4 of Table 4.)

2. For lower I, an unblocked Tukey test that has been corrected via an estimated ρ yields test sizes that can be much more than the nominal test size. (See column 5 of Table 4.)

3. For lower I, and $\rho = 0.5, 0.6$, an unblocked Tukey test that has been corrected via the true ρ yields test sizes that can be lower than the nominal test size. For lower I, and $\rho = 0.8, 0.9, 0.95, 0.99$, an unblocked Tukey test that has been corrected via the true ρ yields test sizes that can be much higher than the nominal test size. (See column 6 of Table 4.)

4. In general, a blocked Tukey test (one that uses an s_{b} denominator) yields actual test sizes that closely match the nominal 0.05 test size. The blocked Tukey test does perform somewhat poorly in the $\rho = 0.95, .99$ cases. (For lower I, actual test sizes fall below the nominal 0.05 value.) (See column 7 of Table 4.)

A listing of the simulation program that produced the size estimates can be found at `http://www1.fpl.fs.fed.us/ps15_tukey_size_sim_code.html`.

5.4 Simultaneous confidence intervals based on Tukey's multiple comparison test

(Here we work with factor 1. The analogous results hold for factors $2, \ldots, F$.)

It is clear from the proof of Theorem 5 that, under predictor sort allocation, in the unblocked case

$$\max_{l_1, l_2 \in \{1,\ldots,K_1\}} \sqrt{IK_2 \ldots K_F} \left| \bar{Y}_{\cdot l_1 \cdot \ldots \cdot} - \bar{Y}_{\cdot l_2 \cdot \ldots \cdot} - (\mu_{l_1 \cdot \ldots \cdot} - \mu_{l_2 \cdot \ldots \cdot}) \right| / s_{\mathrm{ub}} \stackrel{\mathrm{D}}{\to} \sqrt{1-\rho^2} F_{\mathrm{R(K_1)}}$$

and, in the blocked case,

$$\max_{l_1, l_2 \in \{1,\ldots,K_1\}} \sqrt{IK_2 \ldots K_F} \left| \bar{Y}_{\cdot l_1 \cdot \ldots \cdot} - \bar{Y}_{\cdot l_2 \cdot \ldots \cdot} - (\mu_{l_1 \cdot \ldots \cdot} - \mu_{l_2 \cdot \ldots \cdot}) \right| / s_{\mathrm{b}} \stackrel{\mathrm{D}}{\to} F_{\mathrm{R(K_1)}}$$

(where, for example, $\mu_{l_1 \cdot \ldots \cdot}$ is the effect associated with the l_1th level of factor 1). That is, in the unblocked case,

$$\text{Prob}\left(\max_{l_1,l_2\in\{1,\ldots,K_1\}} \sqrt{IK_2\ldots K_F}\left|\bar{Y}_{\cdot l_1\cdot\ldots\cdot} - \bar{Y}_{\cdot l_2\cdot\ldots\cdot} - (\mu_{l_1\cdot\ldots\cdot} - \mu_{l_2\cdot\ldots\cdot})\right|/s_{\text{ub}} \le \sqrt{1-\rho^2}F^{-1}_{\text{R(K}_1)}(1-\alpha)\right) \to 1-\alpha$$

as $I \to \infty$.

So, in the unblocked case, given any $\epsilon > 0$, we can find an N_ϵ such that $I > N_\epsilon$ implies that

$$\begin{aligned}
&\text{Prob}(\bar{Y}_{\cdot l_1\cdot\ldots\cdot} - \bar{Y}_{\cdot l_2\cdot\ldots\cdot} - \left(s_{\text{ub}}/\sqrt{IK_2\ldots K_F}\right)\sqrt{1-\rho^2}F^{-1}_{\text{R(K}_1)}(1-\alpha) \\
\le\ & \mu_{l_1\cdot\ldots\cdot} - \mu_{l_2\cdot\ldots\cdot} \\
\le\ & \bar{Y}_{\cdot l_1\cdot\ldots\cdot} - \bar{Y}_{\cdot l_2\cdot\ldots\cdot} + \left(s_{\text{ub}}/\sqrt{IK_2\ldots K_F}\right)\sqrt{1-\rho^2}F^{-1}_{\text{R(K}_1)}(1-\alpha) \\
&\text{ for all } l_1, l_2 \in \{1,\ldots,K_1\}) > 1-\alpha-\epsilon
\end{aligned}$$

so, simultaneously for all l_1, l_2 pairs in $\{1,\ldots,K_1\}$, for $I > N_\epsilon$,

$$\bar{Y}_{\cdot l_1\cdot\ldots\cdot} - \bar{Y}_{\cdot l_2\cdot\ldots\cdot} \pm \left(s_{\text{ub}}/\sqrt{IK_2\ldots K_F}\right)\sqrt{1-\rho^2}F^{-1}_{\text{R(K}_1)}(1-\alpha)$$

is a $1-\alpha-\epsilon$ confidence interval on $\mu_{l_1\cdot\ldots\cdot} - \mu_{l_2\cdot\ldots\cdot}$.

Similarly, in the blocked case, given any $\epsilon > 0$, we can find an N_ϵ such that for $I > N_\epsilon$, simultaneously for all l_1, l_2 pairs in $\{1,\ldots,K_1\}$,

$$\bar{Y}_{\cdot l_1\cdot\ldots\cdot} - \bar{Y}_{\cdot l_2\cdot\ldots\cdot} \pm \left(s_{\text{b}}/\sqrt{IK_2\ldots K_F}\right)F^{-1}_{\text{R(K}_1)}(1-\alpha)$$

is a $1-\alpha-\epsilon$ confidence interval on $\mu_{l_1\cdot\ldots\cdot} - \mu_{l_2\cdot\ldots\cdot}$.

In practical terms this implies that for I large enough, the

$$\bar{Y}_{\cdot l_1\cdot\ldots\cdot} - \bar{Y}_{\cdot l_2\cdot\ldots\cdot} \pm \left(s_{\text{ub}}/\sqrt{IK_2\ldots K_F}\right)\sqrt{1-\rho^2}F^{-1}_{\text{R(K}_1)}(1-\alpha)$$

intervals are good approximations to $1-\alpha$ simultaneous confidence intervals on the $\mu_{l_1\cdot\ldots\cdot} - \mu_{l_2\cdot\ldots\cdot}$'s.

Similarly, for I large enough, the

$$\bar{Y}_{\cdot l_1\cdot\ldots\cdot} - \bar{Y}_{\cdot l_2\cdot\ldots\cdot} \pm \left(s_{\text{b}}/\sqrt{IK_2\ldots K_F}\right)F^{-1}_{\text{R(K}_1)}(1-\alpha)$$

intervals are good approximations to $1-\alpha$ simultaneous confidence intervals on the $\mu_{l_1\cdot\ldots\cdot} - \mu_{l_2\cdot\ldots\cdot}$'s.

6 Web Programs/R Programs

Forest Products Laboratory scientists have produced predictor sort web programs that help researchers

1. choose sample sizes (perform power calculations) for predictor sort hypothesis tests
2. allocate specimens via a predictor sort
3. perform hypothesis tests for a simple 1 factor, 2 levels predictor sort experiment
4. perform simulations to estimate the coverage of predictor sort confidence intervals on treatment means

These programs can be accessed at `http://www1.fpl.fs.fed.us/predsort.html`. This web page also contains a link to R code that helps users calculate predictor sort confidence intervals on treatment means.

We have also developed interactive Java code that permits a user to obtain small sample confidence intervals on quantiles in the predictor sort case. This work will appear in a separate Forest Products Laboratory technical report.

7 Summary

We have reminded readers that, properly designed and analyzed, predictor sort experiments (experiments in which the predictor variable is used to form blocks) permit scientists to achieve considerable increases in statistical power and/or reductions in sample sizes (and thus reductions in experimental costs). For analysis of variance tests of hypotheses, sample sizes can be reduced from roughly n to $(1-\rho^2)n$ where ρ is the correlation between the predictor/concomitant and the variable of interest. For confidence intervals on quantiles, approximate sample size reductions are illustrated in Figure 40. They can amount to 30% for ρ equal to 0.70, and increase as ρ increases.

Our studies also indicate that the $1-\rho^2$ factor is only approximate, especially for blocked anovas (as opposed to analyses of covariance). Thus, we have provided a web based simulation program that yields estimates of actual powers (in addition to estimates based on large sample theory).

We have demonstrated that if a scientist performs a predictor sort allocation, but then analyzes the experiment as an unblocked analysis of variance, their experiment can have extremely low statistical power (an inability to detect actual differences). This amounts to a serious scientific blunder.

We have demonstrated theoretically that given a predictor sort allocation, unmodified analyses of variance (blocked or unblocked) and analyses of covariance yield incorrect confidence intervals on treatment means. (The confidence intervals are too wide in the unblocked anova case and too narrow in the blocked anova and analysis of covariance cases.) We have provided a web based simulation program that estimates the coverages of incorrect (unmodified) anova confidence intervals on treatment means, and the coverages of corrected anova and anocov confidence intervals. We have also provided an R function that helps a scientist calculate corrected confidence intervals on treatment means estimated from a predictor sort experiment.

We have developed methods that yield correct one-sided lower confidence bounds on quantiles given a predictor sort allocation.

Finally, we have developed Scheffé and Tukey multiple comparison tests and associated simultaneous confidence intervals that are appropriate in the predictor sort case.

All of our results have been established under an assumption of a joint bivariate normal relationship between the predictor and the response.

As noted earlier, it can be argued that in a predictor sort situation a professional statistician would undoubtedly perform a blocked analysis or an analysis of covariance using the predictor/concomitant as the covariate, and thus, we need not exercise special care in identifying, designing, and analyzing predictor sort experiments. We have a five-fold response. First, identifying a design as a predictor-sort design permits a scientist to perform correct power calculations. Second, blocked anova hypothesis tests can perform poorly as ρ becomes sufficiently large. Third, although unmodified blocked anovas (for lower ρ's) and analyses of covariance yield essentially correct hypothesis tests, they yield incorrect confidence intervals on treatment means. Fourth, blocked anovas and analyses of covariance do not help us in the quantile estimation case. Finally, as noted at the end of Section 2, we have seen introductory texts that treat predictor sort allocation as a good experimental practice independent of the method of analysis. (For example, one of the texts we sampled discussed matching, t tests, and ANOVAs, but not paired t-tests, blocked ANOVAS, or analyses of covariance.) Given the large decrease in power (especially for larger ρ's) that can occur if a predictor sort experiment is analyzed via an unblocked anova, this pedagogy must be corrected.

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8 Appendix A — Noncentrality parameter in the hypothesis test simulation

To investigate hypothesis test powers, we needed to choose "informative" noncentrality parameters at which to perform the simulations. When there are J treatments, I "replicates" for each treatment, a correlation ρ between the predictor X and the response Y, and a Y variance of σ_Y^2, the approximately appropriate noncentrality parameter is (see result (5))

$$\sum_{j=1}^{J} I(\mu_j - \bar{\mu}.)^2/((1-\rho^2)\sigma_Y^2)$$

For the purposes of a simulation, σ_Y^2 is just a scale parameter and, without loss of generality, we set it equal to 1 in the simulations. For a correct predictor sort analysis, and "reasonable" I and ρ, we wanted power to increase from .05 to approximately 1 as $\sum_{j=1}^{J}(\mu_j - \bar{\mu}.)^2$ increased from 0 to its maximum simulation value. In figure 2 of Verrill and Green (1996), we saw that for $J = 2$, $I = 24$, and $\rho = .7$, power does increase from 0.05 to approximately 1 as $|\mu_2 - \mu_1|/\sigma_Y$ increases from 0 to 1. For $J = 2$, we have

$$\sum_{j=1}^{J}(\mu_j - \bar{\mu}.)^2 = (\mu_1 - (\mu_1 + \mu_2)/2)^2 + (\mu_2 - (\mu_1 + \mu_2)/2)^2 = (\mu_2 - \mu_1)^2/2$$

Thus if we want to see behavior similar to Verrill and Green's figure 2 behavior for $\rho = .7$ and $I \approx 24$ (and $\sigma_Y = 1$), we would need, for a general J, and power approximately equal to 1,

$$\sum_{j=1}^{J}(\mu_j - \bar{\mu}.)^2 = 1/2 \tag{41}$$

We can obtain this result in an infinite number of ways. We chose to obtain it by providing that, for $J = 2K$, K an integer,

$$\mu_1, \ldots, \mu_J = -K\Delta, \ldots, -\Delta, \Delta, \ldots, K\Delta$$

and for $J = 2K + 1$, K an integer,

$$\mu_1, \ldots, \mu_J = -K\Delta, \ldots, -\Delta, 0, \Delta, \ldots, K\Delta$$

Thus,

$$\sum_{j=1}^{J}(\mu_j - \bar{\mu}.)^2 = 2(K^2\Delta^2 + \ldots + 1^2\Delta^2) = \Delta^2(K)(K+1)(2K+1)/3$$

Setting this equal to 1/2 as suggested by equation (41), we have

$$\Delta = \sqrt{3/(2(K)(K+1)(2K+1))} \tag{42}$$

So, in our simulations we took $\Delta_m = (m-1)\Delta/20$ for Δ given by (42) and $m \in \{1, \ldots, 21\}$. (We refer to m as the "noncentrality parameter index" in Figures 1-32.) This led to powers that varied roughly between 0.05 and 1 as the noncentrality parameters increased in the $I = 20$, $\rho = .7$ case. Powers were, of course, lower for lower I's and ρ's. They were higher for higher I's, and ρ's.

9 Appendix B — An analysis of covariance power calculation

Analyses of covariance have been treated in statistical textbooks for many, many years. See, for example, Scheffé (1959). However, for personal archival purposes, in this appendix we present a derivation of the appropriate sums of squares (and thus non-centrality parameter) in the 1 factor case with one concomitant variable. We used the results of this calculation in the power simulation described in Section 2.

Suppose that we have J "treatments" and I replications for each treatment. We have

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where the elements of $\mathbf{y}$ are $y_{11}, \ldots, y_{I1}, \ldots, y_{1J}, \ldots, y_{IJ}$, the first column of $\mathbf{X}$ contains I ones followed by $I(J-1)$ zeros, ..., the Jth column of $\mathbf{X}$ contains $I(J-1)$ zeros followed by I ones, the elements of the $J+1$th column of $\mathbf{X}$ are $x_{11}, \ldots, x_{I1}, \ldots, x_{1J}, \ldots, x_{IJ}$ where the x_{ij}'s are the values of the concomitant variable,

$$\boldsymbol{\beta} \equiv \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_J \\ b \end{pmatrix}$$

where the μ_j's are the treatment effects and b is the slope associated with the concomitant variable, and

$$\boldsymbol{\epsilon} \sim N(\mathbf{0}, \mathbf{I}_{IJ \times IJ})$$

To proceed with Scheffé's approach, we need $\mathbf{a}_j$, $j = 1, \ldots, J-1$ that lie in the span of the columns of $\mathbf{X}$, and satisfy

$$\begin{aligned} \mathbf{a}_1^T \mathbf{X} &= \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 & 0 \end{pmatrix} \\ &\vdots \\ \mathbf{a}_{J-1}^T \mathbf{X} &= \begin{pmatrix} 1 & 0 & \ldots & 0 & -1 & 0 \end{pmatrix} \end{aligned}$$

It is easy to check that $\mathbf{a}_j$, $j \in \{1, \ldots, J-1\}$, is the sum of two vectors. The first vector has $1/I$ as its first I elements, $-1/I$ as elements $jI+1$ through $(j+1)I$, and zeros elsewhere.

To describe the second vector, we define

$$\bar{x}_{\cdot j} \equiv \sum_{i=1}^{I} x_{ij}/I$$

$$\bar{x}_{\cdot\cdot} \equiv \sum_{j=1}^{J} \sum_{i=1}^{I} x_{ij}/(JI)$$

and

$$D \equiv \sum_{j=1}^{J} \sum_{i=1}^{I} (x_{ij} - \bar{x}_{\cdot j})^2$$

Then the second vector is given by

$$\begin{pmatrix} (x_{11} - \bar{x}_{\cdot 1})(\bar{x}_{\cdot j+1} - \bar{x}_{\cdot 1})/D \\ \vdots \\ (x_{I1} - \bar{x}_{\cdot 1})(\bar{x}_{\cdot j+1} - \bar{x}_{\cdot 1})/D \\ \vdots \\ (x_{1J} - \bar{x}_{\cdot J})(\bar{x}_{\cdot j+1} - \bar{x}_{\cdot 1})/D \\ \vdots \\ (x_{IJ} - \bar{x}_{\cdot J})(\bar{x}_{\cdot j+1} - \bar{x}_{\cdot 1})/D \end{pmatrix}$$

Now let $\mathbf{u}_1, \ldots, \mathbf{u}_{J-1}$ denote an orthonormal basis of the linear span of $\mathbf{a}_1, \ldots, \mathbf{a}_{J-1}$. The numerator sum of squares of a test of the hypothesis that $\mu_1 = \mu_2 = \ldots = \mu_J$ is

$$\mathrm{SS}_{\mathrm{num}} = \sum_{j=1}^{J-1} (\mathbf{u}_j^T \mathbf{y})^2 \tag{43}$$

It is clear that the $\mathbf{a}_j$'s are perpendicular to $\mathbf{1}$, the vector of JI 1's, and to the $J+1$th column of $\boldsymbol{X}$. We denote this column by $\boldsymbol{x}$. Thus it is clear that $\mathbf{u}_1, \ldots, \mathbf{u}_{J-1}$, $\mathbf{1}/\sqrt{JI}$, $(\boldsymbol{x} - \mathbf{1}\bar{x}_{\cdot\cdot})/\sqrt{\sum_{j=1}^J \sum_{i=1}^I (x_{ij} - \bar{x}_{\cdot\cdot})^2}$ form an orthonormal basis of the linear span of the columns of $\mathbf{X}$.

Another orthonormal basis of the linear span of the columns of $\mathbf{X}$ is $\mathbf{w}_1, \ldots, \mathbf{w}_{J+1}$ where for $j = 1, \ldots, J$, $\mathbf{w}_j$ has the value $1/\sqrt{I}$ for the $(j-1)I+1, \ldots, jI$ components and 0 elsewhere, and

$$\mathbf{w}_{J+1} = \begin{pmatrix} (x_{11} - \bar{x}_{\cdot 1})/\sqrt{D} \\ \vdots \\ (x_{I1} - \bar{x}_{\cdot 1})/\sqrt{D} \\ \vdots \\ (x_{1J} - \bar{x}_{\cdot J})/\sqrt{D} \\ \vdots \\ (x_{IJ} - \bar{x}_{\cdot J})/\sqrt{D} \end{pmatrix}$$

Thus,

$$\sum_{j=1}^{J+1} (\mathbf{u}_j^T \mathbf{y})^2 = \sum_{j=1}^{J+1} (\mathbf{w}_j^T \mathbf{y})^2 \tag{44}$$

From results (43) and (44), we have

$$\begin{aligned}
\mathrm{SS}_{\mathrm{num}} &= \sum_{j=1}^{J+1} (\mathbf{w}_j^T \mathbf{y})^2 \\
&\quad - \left(\mathbf{y}^T \mathbf{1}/\sqrt{JI}\right)^2 \\
&\quad - \left(\mathbf{y}^T (\boldsymbol{x} - \mathbf{1}\bar{x}_{\cdot\cdot})/\sqrt{\sum_{j=1}^J \sum_{i=1}^I (x_{ij} - \bar{x}_{\cdot\cdot})^2}\right)^2 \\
&= \sum_{j=1}^J I\bar{y}_{\cdot j}^2 + \left(\sum_{j=1}^J \sum_{i=1}^I (y_{ij} - \bar{y}_{\cdot j})(x_{ij} - \bar{x}_{\cdot j})\right)^2 / \sum_{j=1}^J \sum_{i=1}^I (x_{ij} - \bar{x}_{\cdot j})^2 - IJ\bar{y}_{\cdot\cdot}^2 \\
&\quad - \left(\sum_{j=1}^J \sum_{i=1}^I (y_{ij} - \bar{y}_{\cdot\cdot})(x_{ij} - \bar{x}_{\cdot\cdot})\right)^2 / \sum_{j=1}^J \sum_{i=1}^I (x_{ij} - \bar{x}_{\cdot\cdot})^2 \\
&= \sum_{j=1}^J I(\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})^2 + \left(\sum_{j=1}^J \sum_{i=1}^I (y_{ij} - \bar{y}_{\cdot j})(x_{ij} - \bar{x}_{\cdot j})\right)^2 / \sum_{j=1}^J \sum_{i=1}^I (x_{ij} - \bar{x}_{\cdot j})^2 \\
&\quad - \left(\sum_{j=1}^J \sum_{i=1}^I (y_{ij} - \bar{y}_{\cdot\cdot})(x_{ij} - \bar{x}_{\cdot\cdot})\right)^2 / \sum_{j=1}^J \sum_{i=1}^I (x_{ij} - \bar{x}_{\cdot\cdot})^2
\end{aligned} \tag{45}$$

Now we know that we can obtain the noncentrality parameter, δ, for performing power calculations by replacing $\mathbf{y}$ in the numerator sum of squares by $\mathbf{X}\boldsymbol{\beta}$ and dividing the result by σ^2. That is, in the current case

$$\begin{array}{c} y_{ij} \text{ is replaced by } \mu_j + b x_{ij} \\ \bar{y}_{\cdot j} \text{ is replaced by } \mu_j + b\bar{x}_{\cdot j} \\ \bar{y}_{\cdot\cdot} \text{ is replaced by } \bar{\mu}_{\cdot} + b\bar{x}_{\cdot\cdot} \end{array}$$

Applying this to result (45), we have

$$\begin{aligned} \sigma^2 \times \delta &= \sum_{j=1}^{J} I\left(\mu_j - \bar{\mu}_{\cdot} + b(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})\right)^2 + \left(\sum_{j=1}^{J}\sum_{i=1}^{I} b(x_{ij} - \bar{x}_{\cdot j})^2\right)^2 / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{ij} - \bar{x}_{\cdot j})^2 \\ &\quad - \left(\sum_{j=1}^{J}\sum_{i=1}^{I}\left(\mu_j - \bar{\mu}_{\cdot} + b(x_{ij} - \bar{x}_{\cdot\cdot})\right)(x_{ij} - \bar{x}_{\cdot\cdot})\right)^2 / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{ij} - \bar{x}_{\cdot\cdot})^2 \\ &= \sum_{j=1}^{J} I(\mu_j - \bar{\mu}_{\cdot})^2 + 2b\sum_{j=1}^{J} I(\mu_j - \bar{\mu}_{\cdot})(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}) + b^2 \sum_{j=1}^{J} I(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})^2 \\ &\quad + b^2 \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{ij} - \bar{x}_{\cdot j})^2 \\ &\quad - \left(\sum_{j=1}^{J} I(\mu_j - \bar{\mu}_{\cdot})(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}) + b\sum_{j=1}^{J}\sum_{i=1}^{I}(x_{ij} - \bar{x}_{\cdot\cdot})^2\right)^2 / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{ij} - \bar{x}_{\cdot\cdot})^2 \qquad (46) \end{aligned}$$

Expanding the last term in (46) and cancelling, we obtain

$$\sigma^2 \times \delta = \sum_{j=1}^{J} I(\mu_j - \bar{\mu}_{\cdot})^2 - \left(\sum_{j=1}^{J} I(\mu_j - \bar{\mu}_{\cdot})(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})\right)^2 / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{ij} - \bar{x}_{\cdot\cdot})^2 \qquad (47)$$

So, by the Cauchy-Schwarz theorem,

$$\begin{aligned} \sigma^2 \times \delta &\geq \sum_{j=1}^{J} I(\mu_j - \bar{\mu}_{\cdot})^2 - \left(\sum_{j=1}^{J} I(\mu_j - \bar{\mu}_{\cdot})^2 \sum_{j=1}^{J} I(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})^2\right) / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{ij} - \bar{x}_{\cdot\cdot})^2 \\ &= \sum_{j=1}^{J} I(\mu_j - \bar{\mu}_{\cdot})^2 \left(1 - \sum_{j=1}^{J} I(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})^2 / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{ij} - \bar{x}_{\cdot\cdot})^2\right) \end{aligned}$$

Note that we can make this lower bound on the noncentrality parameter larger by minimizing

$$\sum_{j=1}^{J}(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})^2$$

That is, by making the $\bar{x}_{\cdot j}$ as similar as possible. A predictor sort allocation helps to do this. Thus, even if one intends to perform an analysis of covariance, a predictor sort allocation can improve performance.

Because $\mathbf{w}_1, \ldots, \mathbf{w}_{J+1}$ form an orthonormal basis for the linear span of the columns of $\mathbf{X}$, the denominator sum of squares for a test of the hypothesis that $\mu_1 = \mu_2 = \ldots = \mu_J$ is

$$\begin{aligned}
\mathrm{SS}_{\mathrm{den}} &= \mathbf{y}^T\mathbf{y} - \sum_{j=1}^{J+1}(\mathbf{w}_j^T\mathbf{y})^2 \\
&= \mathbf{y}^T\mathbf{y} - \sum_{j=1}^{J} I\bar{y}_{\cdot j}^2 \\
&\quad - \left(\sum_{j=1}^{J}\sum_{i=1}^{I}(y_{ij}-\bar{y}_{\cdot j})(x_{ij}-\bar{x}_{\cdot j})\right)^2 / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{ij}-\bar{x}_{\cdot j})^2 \\
&= \sum_{j=1}^{J}\sum_{i=1}^{I}(y_{ij}-\bar{y}_{\cdot j})^2 - \left(\sum_{j=1}^{J}\left(\sum_{i=1}^{I}x_{ij}y_{ij} - I\bar{x}_{\cdot j}\bar{y}_{\cdot j}\right)\right)^2 / \sum_{j=1}^{J}\left(\sum_{i=1}^{I}x_{ij}^2 - I\bar{x}_{\cdot j}^2\right) \qquad (48)
\end{aligned}$$

From equations (45) and (48), we can calculate the F statistic for testing the hypothesis that $\mu_1 = \mu_2 = \ldots = \mu_J$:

$$F_{\mathrm{anocov}} = \left(\frac{\mathrm{SS}_{\mathrm{num}}}{J-1}\right) / \left(\frac{\mathrm{SS}_{\mathrm{den}}}{JI-(J+1)}\right)$$

The power of this test is

$$\mathrm{Power}_{\mathrm{anocov}} = 1 - \mathrm{NCF}_{J-1,JI-(J+1)}(\delta)(\mathrm{F}^{-1}_{J-1,JI-(J+1)}(1-\alpha)) \qquad (49)$$

where the noncentrality parameter, δ, is given by equation (47), α is the significance level, $\mathrm{NCF}_{J-1,JI-(J+1)}(\delta)(s)$ is the noncentral F distribution function (evaluated at s) with noncentrality parameter δ, $J-1$ numerator degrees of freedom, and $JI-(J+1)$ denominator degrees of freedom, and $\mathrm{F}^{-1}_{J-1,JI-(J+1)}$ is the inverse distribution function of a central F distribution with $J-1$ numerator and $JI-(J+1)$ denominator degrees of freedom.

10 Appendix C — Relations between analysis of covariance and maximum likelihood estimators

Given that least squares and maximum likelihood estimators are essentially equivalent (up to divisors for estimates of variances) for linear models with Gaussian errors, we would expect near equivalence for analysis of covariance and maximum likelihood estimators. This expectation is somewhat clouded by the fact that the probability models differ. In the maximum likelihood model we treat the x values as random variables that are correlated with the response values. In the analysis of covariance model, we treat the x's as fixed constants. (As noted in Section 3, this can lead a statistician/scientist to calculate an analysis of covariance confidence interval on a treatment effect that is too narrow.)

In this appendix we establish that the two approaches yield essentially equivalent estimates and this, of course, then yields (MLE) confidence intervals for the analysis of covariance estimators. Here we restrict our attention to an analysis of covariance with one factor and one predictor. We assume that there are J treatments and I observations per treatment.

The MLE is based on the following model:

$$
\begin{aligned}
y_{ij} &= \mu_j + \sigma_Y \left(\rho(x_{ij} - \mu_X)/\sigma_X + \sqrt{1-\rho^2} z_{ij} \right) \\
&= \mu_j - \rho(\sigma_Y/\sigma_X)\mu_X + \rho(\sigma_Y/\sigma_X)\, x_{ij} + \sqrt{1-\rho^2}\, \sigma_Y\, z_{ij} \\
&= a_j + b\, x_{ij} + \sqrt{1-\rho^2}\, \sigma_Y\, z_{ij}
\end{aligned}
$$

where the x's are iid $\mathrm{N}(\mu_X, \sigma_X^2)$, the z's are iid $\mathrm{N}(0,1)$, and the x's and z's are independent. (This is in the standard, non predictor sort case.)

Thus, we have the identities

$$
a_j = \mu_j - \rho(\sigma_Y/\sigma_X)\mu_X \tag{50}
$$

$$
b = \rho(\sigma_Y/\sigma_X) \tag{51}
$$

and

$$
\sigma = \sqrt{1-\rho^2}\, \sigma_Y \tag{52}
$$

where $a_1, \ldots, a_J$, b, and σ are the parameters of the anocov model, and $\mu_1, \ldots, \mu_J$, μ_X, ρ, σ_X, and σ_Y are the parameters of the MLE model.

We know that the anocov estimator of b is $\boldsymbol{y}^{\mathrm{T}} \boldsymbol{v}_b$ where

$$
\boldsymbol{v}_b = \begin{pmatrix} (x_{11} - \bar{x}_{\cdot 1})/D \\ \vdots \\ (x_{I1} - \bar{x}_{\cdot 1})/D \\ \vdots \\ (x_{1J} - \bar{x}_{\cdot J})/D \\ \vdots \\ (x_{IJ} - \bar{x}_{\cdot J})/D \end{pmatrix}
$$

and

$$
D \equiv \sum_{j=1}^{J} \sum_{i=1}^{I} (x_{ij} - \bar{x}_{\cdot j})^2
$$

so

$$
\hat{b} = \frac{\sum_{j=1}^{J} \sum_{i=1}^{I} (y_{ij} - \bar{y}_{\cdot j})(x_{ij} - \bar{x}_{\cdot j})}{\sum_{j=1}^{J} \sum_{i=1}^{I} (x_{ij} - \bar{x}_{\cdot j})^2} \tag{53}
$$

We also know that the anocov estimator of a_j is $\boldsymbol{y}^{\mathrm{T}} \boldsymbol{v}_j$ where $\boldsymbol{v}_j$ is a sum of two vectors. The first vector has $1/I$ in rows $(j-1)I+1, \ldots, jI$ and zeros elsewhere. The second vector equals

$$
-\boldsymbol{v}_b \bar{x}_{\cdot j}
$$

so

$$
\hat{a}_j = \bar{y}_{\cdot j} - \hat{b}\bar{x}_{\cdot j} \tag{54}
$$

Finally, from (48), we know that we can calculate $\hat{\sigma}^2$ via

$$
(IJ - (J+1))\hat{\sigma}^2 = \sum_{j=1}^{J} \sum_{i=1}^{I} y_{ij}^2 - \sum_{j=1}^{J} I \bar{y}_{\cdot j}^2 - \left(\sum_{j=1}^{J} \sum_{i=1}^{I} (y_{ij} - \bar{y}_{\cdot j})(x_{ij} - \bar{x}_{\cdot j}) \right)^2 / \sum_{j=1}^{J} \sum_{i=1}^{I} (x_{ij} - \bar{x}_{\cdot j})^2 \tag{55}
$$

Now, formally, there is no μ_X parameter in the analysis of covariance model. However, given equations (50) and (51), it is reasonable to take

$$\hat{a}_j + \hat{b}\bar{x}_{\cdot\cdot}$$

as the analysis of covariance estimator of μ_j. Given equation (54) this implies that

$$\hat{\mu}_{\text{anocov},j} \equiv \hat{a}_j + \hat{b}\bar{x}_{\cdot\cdot} = \bar{y}_{\cdot j} - \hat{b}(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}) \tag{56}$$

is our anocov estimator of μ_j. From appendix A.2 of Verrill *et al.* (2004), we know that the maximum likelihood estimator of μ_j is

$$\hat{\mu}_{\text{ml},j} \equiv \bar{y}_{\cdot j} - \hat{\rho}(\hat{\sigma}_Y/\hat{\sigma}_X)(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}) \tag{57}$$

where $\hat{\rho}$, $\hat{\sigma}_Y$, and $\hat{\sigma}_X$ are the maximum likelihood estimators of ρ, σ_Y, and σ_X. Thus, to establish the equivalence of the anocov and MLE estimators of μ_j, we need to show that

$$\hat{b} = \hat{\rho}(\hat{\sigma}_Y/\hat{\sigma}_X)$$

From results (36), (38), and (39) in Verrill *et al.* (2004) we have

$$\begin{aligned}
\hat{\rho}(\hat{\sigma}_Y/\hat{\sigma}_X) &= \frac{\sum_{j=1}^J \sum_{i=1}^I (x_{ij} - \bar{x}_{\cdot j})(y_{ij} - \bar{y}_{\cdot j})}{\sum_{j=1}^J \sum_{i=1}^I (x_{ij} - \bar{x}_{\cdot\cdot})^2 - I\sum_{j=1}^J (\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})^2} \\
&= \frac{\sum_{j=1}^J \sum_{i=1}^I (x_{ij} - \bar{x}_{\cdot j})(y_{ij} - \bar{y}_{\cdot j})}{\sum_{j=1}^J \sum_{i=1}^I (x_{ij} - \bar{x}_{\cdot j})^2} \\
&= \hat{b}
\end{aligned}$$

We can also establish the relation between the anocov $\hat{\sigma}^2$ and the MLE $\hat{\sigma}_Y^2$. From results (26), (28), (36), and (37) in Verrill *et al.* (2004) we have

$$\begin{aligned}
\hat{\sigma}_Y^2 &= \sum_{j=1}^J \sum_{i=1}^I \left(y_{ij} - (\bar{y}_{\cdot j} - \hat{\rho}(\hat{\sigma}_Y/\hat{\sigma}_X)(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}))\right)^2/(IJ) \\
&= \sum_{j=1}^J \sum_{i=1}^I \left(y_{ij} - \bar{y}_{\cdot j} + \hat{b}(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})\right)^2/(IJ) \\
&= \left(\sum_{j=1}^J \sum_{i=1}^I (y_{ij} - \bar{y}_{\cdot j})^2 + \hat{b}^2 \sum_{j=1}^J I(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})^2\right)/(IJ) \\
&= \left(\sum_{j=1}^J \sum_{i=1}^I (y_{ij} - \bar{y}_{\cdot j})^2 + \frac{\hat{\rho}^2\hat{\sigma}_Y^2}{\hat{\sigma}_X^2} \sum_{j=1}^J I(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})^2\right)/(IJ)
\end{aligned}$$

so

$$IJ\hat{\sigma}_Y^2 - IJ\hat{\rho}^2\hat{\sigma_Y}^2 \frac{\sum_{i=1}^I I(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})^2}{\sum_{j=1}^J \sum_{i=1}^I (x_{ij} - \bar{x}_{\cdot\cdot})^2} = \sum_{j=1}^J \sum_{i=1}^I (y_{ij} - \bar{y}_{\cdot j})^2$$

or

$$IJ\hat{\sigma}_Y^2 \left(1 - \hat{\rho}^2 \frac{\sum_{i=1}^I I(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})^2}{\sum_{j=1}^J \sum_{i=1}^I (x_{ij} - \bar{x}_{\cdot j})^2 + \sum_{j=1}^J I(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot})^2}\right) = \sum_{j=1}^J \sum_{i=1}^I (y_{ij} - \bar{y}_{\cdot j})^2$$

Thus

$$IJ\hat{\sigma}_Y^2\left(1-\hat{\rho}^2+\hat{\rho}^2-\hat{\rho}^2\frac{\sum_{i=1}^I I(\bar{x}_{\cdot j}-\bar{x}_{\cdot\cdot})^2}{\sum_{j=1}^J\sum_{i=1}^I(x_{ij}-\bar{x}_{\cdot j})^2+\sum_{j=1}^J I(\bar{x}_{\cdot j}-\bar{x}_{\cdot\cdot})^2}\right)=\sum_{j=1}^J\sum_{i=1}^I(y_{ij}-\bar{y}_{\cdot j})^2$$

and

$$\hat{\sigma}_Y^2(1-\hat{\rho}^2)+\hat{\sigma}_Y^2\hat{\rho}^2\left(1-\frac{\sum_{i=1}^I I(\bar{x}_{\cdot j}-\bar{x}_{\cdot\cdot})^2}{\sum_{j=1}^J\sum_{i=1}^I(x_{ij}-\bar{x}_{\cdot j})^2+\sum_{j=1}^J I(\bar{x}_{\cdot j}-\bar{x}_{\cdot\cdot})^2}\right)=\sum_{j=1}^J\sum_{i=1}^I(y_{ij}-\bar{y}_{\cdot j})^2/(IJ)$$

or

$$\hat{\sigma}_Y^2(1-\hat{\rho}^2)+\hat{\sigma}_Y^2\hat{\rho}^2\left(\frac{\sum_{j=1}^J\sum_{i=1}^I(x_{ij}-\bar{x}_{\cdot j})^2}{\sum_{j=1}^J\sum_{i=1}^I(x_{ij}-\bar{x}_{\cdot j})^2+\sum_{j=1}^J I(\bar{x}_{\cdot j}-\bar{x}_{\cdot\cdot})^2}\right)=\sum_{j=1}^J\sum_{i=1}^I(y_{ij}-\bar{y}_{\cdot j})^2/(IJ)$$

so

$$\begin{aligned}\hat{\sigma}_Y^2(1-\hat{\rho}^2) &= \sum_{j=1}^J\sum_{i=1}^I(y_{ij}-\bar{y}_{\cdot j})^2/(IJ)-\frac{\hat{\sigma}_Y^2\hat{\rho}^2}{\hat{\sigma}_X^2}\left(\frac{\sum_{j=1}^J\sum_{i=1}^I(x_{ij}-\bar{x}_{\cdot j})^2}{IJ}\right)\\ &= \sum_{j=1}^J\sum_{i=1}^I(y_{ij}-\bar{y}_{\cdot j})^2/(IJ)-\hat{b}^2\left(\frac{\sum_{j=1}^J\sum_{i=1}^I(x_{ij}-\bar{x}_{\cdot j})^2}{IJ}\right) \qquad (58)\end{aligned}$$

Thus, by results (53), (55), and (58)

$$\begin{aligned}\hat{\sigma}_Y^2(1-\hat{\rho}^2) &= \sum_{j=1}^J\sum_{i=1}^I(y_{ij}-\bar{y}_{\cdot j})^2/(IJ)-\left(\frac{\sum_{j=1}^J\sum_{i=1}^I(y_{ij}-\bar{y}_{\cdot j})(x_{ij}-\bar{x}_{\cdot j})}{\sum_{j=1}^J\sum_{i=1}^I(x_{ij}-\bar{x}_{\cdot j})^2}\right)^2\frac{\sum_{j=1}^J\sum_{i=1}^I(x_{ij}-\bar{x}_{\cdot j})^2}{IJ}\\ &= \left(\sum_{j=1}^J\sum_{i=1}^I(y_{ij}-\bar{y}_{\cdot j})^2-\frac{\left(\sum_{j=1}^J\sum_{i=1}^I(y_{ij}-\bar{y}_{\cdot j})(x_{ij}-\bar{x}_{\cdot j})\right)^2}{\sum_{j=1}^J\sum_{i=1}^I(x_{ij}-\bar{x}_{\cdot j})^2}\right)/(IJ)\\ &= \hat{\sigma}^2(IJ-(J+1))/(IJ)\end{aligned}$$

11 Appendix D — A maximum likelihood variance calculation

One can show (see appendix A of Verrill *et al.* (2004) and, for example, Theorem 6.1 of Chapter 6 of Lehmann (1983)) that when everything is independent, and there are J "treatments" and I bivariate normal X_{ij}, Y_{ij} pairs for each treatment such that, for the jth treatment,

$$\begin{pmatrix} X_{ij} \\ Y_{ij}\end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix}\mu_X\\ \mu_j\end{pmatrix},\begin{pmatrix}\sigma_X^2 & \rho\sigma_X\sigma_Y\\ \rho\sigma_X\sigma_Y & \sigma_Y^2\end{pmatrix}\right)$$

then the maximum likelihood information matrix for $\hat{\mu}_X,\hat{\mu}_1,\ldots,\hat{\mu}_J,\hat{\rho},\hat{\sigma}_X$, $\hat{\sigma}_Y$ is given by

$$\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}^{\mathrm{T}}\\ \boldsymbol{B} & \boldsymbol{C}\end{pmatrix}$$

where $\boldsymbol{A}$ is a $J+1$ by $J+1$ matrix, $\boldsymbol{B}$ is a 3 by $J+1$ matrix of zeros, and $\boldsymbol{C}$ is a 3 by 3 matrix. Thus, we can obtain the asymptotic covariance matrix of $\hat{\mu}_X,\hat{\mu}_1,\ldots,\hat{\mu}_J$ by inverting $\boldsymbol{A}$. Let the

ijth element of $\boldsymbol{A}$ be denoted by a_{ij}. Verrill *et al.* (2004) (also see Theorem 6.1 of Chapter 6 of Lehmann (1983) for the construction of the information matrix when there are J samples from distinct distributions) show that

$$a_{11} = 1/((1-\rho^2)\sigma_X^2)$$

and, for $j \in \{2, \ldots, J+1\}$,

$$a_{jj} = 1/(J(1-\rho^2)\sigma_Y^2)$$

and

$$a_{j1} = a_{1j} = -\rho/(J(1-\rho^2)\sigma_X\sigma_Y)$$

The remaining elements of $\boldsymbol{A}$ are zeros. Using well-known blockwise inversion formulas (see, for example, problem 2.7 in section 1b of Rao (1973)), one can obtain the inverse of $\boldsymbol{A}$. Alternatively, one can begin with a 3 by 3 $\boldsymbol{A}$ ($J = 2$), go to a 4 by 4 $\boldsymbol{A}$ ($J = 3$), and quickly see a pattern. In fact, one can check that if $\boldsymbol{A}$ has $a_{11} = c$, $a_{j1} = a_{1j} = b$ for $j \in \{2, \ldots, J+1\}$, $a_{jj} = a$ for $j \in \{2, \ldots, J+1\}$, and $a_{ij} = 0$ elsewhere, then

$$\boldsymbol{A}^{-1} = B/(ca^2 - Jab^2)$$

where $b_{11} = a^2$, $b_{j1} = b_{1j} = -ba$ for $j \in \{2, \ldots, J+1\}$, $b_{jj} = ca - (J-1)b^2$ for $j \in \{2, \ldots, J+1\}$, and $b_{ij} = b^2$ elsewhere.

Applying this result to the information matrix, we have (assuming that we can verify the conditions of Lehmann's chapter 6 theorem 6.1)

$$\sqrt{IJ}(\hat{\mu}_{\mathrm{ml},j} - \mu_j) \xrightarrow{\mathrm{D}} \mathrm{N}(0, J\sigma_Y^2(1-\rho^2+\rho^2/J))$$

or

$$\sqrt{I}(\hat{\mu}_{\mathrm{ml},j} - \mu_j) \xrightarrow{\mathrm{D}} \mathrm{N}(0, \sigma_Y^2(1-\rho^2+\rho^2/J)) \tag{59}$$

where $\hat{\mu}_{\mathrm{ml},j}$ is given by (57) in Appendix C. That is, the asymptotic variance of the maximum likelihood estimate of μ_j is the value that we found for $\bar{Y}_{\cdot j}$ in the predictor sort case.

12 Appendix E — Asymptotic distribution of the analysis of covariance estimator of μ_j after a predictor sort allocation

As noted in Section 3, Verrill (1999) established that

$$\sqrt{I}(\bar{y}_{\cdot j} - \mu_j) \xrightarrow{\mathrm{D}} \mathrm{N}(0, \sigma_Y^2(1-\rho^2+\rho^2/J)) \tag{60}$$

in the J treatments, I blocks predictor sort case.

Our claim is that in this case, we also have (see result (56))

$$\sqrt{I}\left(\hat{\mu}_{\mathrm{anocov},j} - \mu_j\right) = \sqrt{I}\left(\bar{y}_{\cdot j} - \hat{b}(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}) - \mu_j\right) \xrightarrow{\mathrm{D}} \mathrm{N}\left(0, \sigma_Y^2(1-\rho^2+\rho^2/J)\right) \tag{61}$$

where $\hat{b}$ is the standard analysis of covariance estimate of b in the model

$$y_{ij} = a_j + b \times x_{ij} + \epsilon_{ij}$$

Given (60), we can establish (61) by showing that, given a predictor sort allocation,

$$\sqrt{I}\,\hat{b}\,(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}) \xrightarrow{\mathrm{P}} 0$$

as $I \to \infty$. In the notation introduced at the beginning of Section 2

$$\sqrt{I}\,(\bar{x}_{\cdot j} - \bar{x}_{\cdot\cdot}) = \sqrt{I}\,\left(\bar{x}_{k(\cdot,j),n} - \bar{x}_{k(\cdot,\cdot),n}\right)$$

which converges in probability to 0 as $I \to \infty$ by work done in the proof of (A.1) in the appendix to Verrill (1993). Thus, we need only establish that $\hat{b} = O_p(1)$ to complete the proof of the claim.

We know that, in standard anocov notation,

$$\hat{b} = \sum_{j=1}^{J}\sum_{i=1}^{I}(y_{ij} - \bar{y}_{\cdot j})(x_{ij} - \bar{x}_{\cdot j}) / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{ij} - \bar{x}_{\cdot j})^2$$

In the predictor sort notation of equation (2) this becomes

$$\hat{b} = \sigma_Y \rho/\sigma_X + \sigma_Y\sqrt{1-\rho^2}\sum_{j=1}^{J}\sum_{i=1}^{I}(p_{ij} - \bar{p}_{\cdot j})(x_{k(i,j),n} - \bar{x}_{k(\cdot,j),n}) / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{k(i,j),n} - \bar{x}_{k(\cdot,j),n})^2 \quad (62)$$

By the Cauchy-Schwarz theorem we have

$$\begin{aligned}
&\sum_{j=1}^{J}\sum_{i=1}^{I}(p_{ij} - \bar{p}_{\cdot j})(x_{k(i,j),n} - \bar{x}_{k(\cdot,j),n}) / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{k(i,j),n} - \bar{x}_{k(\cdot,j),n})^2 \\
\le\ & \left(\sum_{j=1}^{J}\sum_{i=1}^{I}(p_{ij} - \bar{p}_{\cdot j})^2\right)^{1/2} \left(\sum_{j=1}^{J}\sum_{i=1}^{I}(x_{k(i,j),n} - \bar{x}_{k(\cdot,j),n})^2\right)^{1/2} / \sum_{j=1}^{J}\sum_{i=1}^{I}(x_{k(i,j),n} - \bar{x}_{k(\cdot,j),n})^2 \\
=\ & \left(\frac{\sum_{j=1}^{J}\sum_{i=1}^{I}(p_{ij} - \bar{p}_{\cdot j})^2/(JI)}{\sum_{j=1}^{J}\sum_{i=1}^{I}(x_{k(i,j),n} - \bar{x}_{k(\cdot,j),n})^2/(JI)}\right)^{1/2} \equiv (N/D)^{1/2} \qquad (63)
\end{aligned}$$

We know that

$$N \xrightarrow{p} 1 \qquad (64)$$

as $I \to \infty$. Also,

$$D = \sum_{j=1}^{J}\sum_{i=1}^{I} x^2_{k(i,j),n}/(JI) - \sum_{j=1}^{J}\bar{x}^2_{k(\cdot,j),n}/J \qquad (65)$$

We know that

$$\sum_{j=1}^{J}\sum_{i=1}^{I} x^2_{k(i,j),n}/(JI) = \sum_{j=1}^{J}\sum_{i=1}^{I} x^2_{ij}/(JI) \xrightarrow{p} \mu_X^2 + \sigma_X^2 \qquad (66)$$

as $I \to \infty$. Also, for all $j \in \{1, \ldots, J\}$,

$$\begin{aligned}
|\bar{x}_{k(\cdot,j),n} - \bar{x}_{\cdot\cdot}| &= |\bar{x}_{k(\cdot,j),n} - \bar{x}_{k(\cdot,\cdot),n}| \\
&= |\sum_{i=1}^{I}(x_{k(i,j),n} - \bar{x}_{k(i,\cdot),n})/I| \\
&\le \sum_{i=1}^{I}|x_{k(i,j),n} - \bar{x}_{k(i,\cdot),n}|/I \\
&\le \sum_{i=1}^{I}(x_{iJ,n} - x_{(i-1)J+1,n})/I \\
&\le (x_{n,n} - x_{1,n})/I \xrightarrow{p} 0
\end{aligned}$$

as $I \to \infty$ by the lemma proven in the Appendix to Verrill (1993). Together with the fact that

$$\bar{x}_{\cdot\cdot} \xrightarrow{p} \mu_X$$

this yields

$$\sum_{j=1}^{J} \bar{x}^2_{k(\cdot,j),n}/J \xrightarrow{p} \mu^2_X \tag{67}$$

Results (65), (66), and (67) yield

$$D \xrightarrow{p} \sigma^2_X \tag{68}$$

as $I \to \infty$. Results (62), (63), (64), and (68) establish that $\hat{b} = O_p(1)$ which in turn completes the proof of result (61).

13 Appendix F — Proof of the Tukey multiple comparison theorem (Theorem 5)

Here we are using the notation introduced at the beginning of Section 5. We work with the first factor, but the same proof will hold for all F factors.

Consider the Tukey numerator

$$Q_I \equiv \max_{l_1,l_2\in\{1,\ldots,K_1\}} \sqrt{IK_2\ldots K_F}|\bar{Y}_{\cdot l_1\cdot\ldots\cdot} - \bar{Y}_{\cdot l_2\cdot\ldots\cdot}|$$

Under the null hypothesis that $\mu_{1\cdot\ldots\cdot} = \ldots = \mu_{K_1\cdot\ldots\cdot}$, we have

$$\begin{aligned} Q_I &= \max_{l_1,l_2\in\{1,\ldots,K_1\}} \sqrt{IK_2\ldots K_F}\left|\rho\bar{X}_{k(\cdot l_1\cdot\ldots\cdot),n} + \sqrt{1-\rho^2}\bar{P}_{\cdot l_1\cdot\ldots\cdot} - \left(\rho\bar{X}_{k(\cdot l_2\cdot\ldots\cdot),n} + \sqrt{1-\rho^2}\bar{P}_{\cdot l_2\cdot\ldots\cdot}\right)\right|\sigma_Y \\ &= \max_{l_1,l_2\in\{1,\ldots,K_1\}} \left|\rho\sqrt{IK_2\ldots K_F}\left(\bar{X}_{k(\cdot l_1\cdot\ldots\cdot),n} - \bar{X}_{k(\cdot l_2\cdot\ldots\cdot),n}\right) + \sqrt{1-\rho^2}\sqrt{IK_2\ldots K_F}\left(\bar{P}_{\cdot l_1\cdot\ldots\cdot} - \bar{P}_{\cdot l_2\cdot\ldots\cdot}\right)\right|\sigma_Y \\ &= \max_{l_1,l_2\in\{1,\ldots,K_1\}} |\Delta_{I,l_1,l_2} + (N_{I,l_1} - N_{I,l_2})|\,\sigma_Y\sqrt{1-\rho^2} \end{aligned} \tag{69}$$

where

$$\Delta_{I,l_1,l_2} \equiv \rho\sqrt{IK_2\ldots K_F}\left(\bar{X}_{k(\cdot l_1\cdot\ldots\cdot),n} - \bar{X}_{k(\cdot l_2\cdot\ldots\cdot),n}\right)/\sqrt{1-\rho^2}$$

and the $N_{I,l}$'s are independent N(0,1)'s that are also independent of the Δ_{I,l_1,l_2}'s.

Now, under predictor sort allocation, for all $l_1, l_2 \in \{1,\ldots,K_1\}$,

$$\begin{aligned} \sqrt{I}\left|\bar{X}_{k(\cdot l_1\cdot\ldots\cdot),n} - \bar{X}_{k(\cdot l_2\cdot\ldots\cdot),n}\right| &\le \sum_{i=1}^{I}\left|\bar{X}_{k(il_1\cdot\ldots\cdot),n} - \bar{X}_{k(il_2\cdot\ldots\cdot),n}\right|/\sqrt{I} \\ &\le \sum_{i=1}^{I}|\max X \text{ in block } i - \min X \text{ in block } i|/\sqrt{I} \\ &\le (\max \text{ overall } X - \min \text{ overall } X)/\sqrt{I} \end{aligned}$$

which converges in probability to zero by the lemma proven in the appendix of Verrill (1993).

Thus, for all l_1, l_2 pairs,

$$\Delta_{I,l_1,l_2} \xrightarrow{\text{p}} 0$$

as $I \to \infty$, so

$$\max_{l_1,l_2 \in \{1,\ldots,K_1\}} |\Delta_{I,l_1,l_2}| \xrightarrow{\text{p}} 0 \tag{70}$$

Next, for all $l_1, l_2 \in \{1,\ldots,K_1\}$,

$$\begin{aligned} |\Delta_{I,l_1,l_2} + (N_{I,l_1} - N_{I,l_2})| &\leq |\Delta_{I,l_1,l_2}| + |N_{I,l_1} - N_{I,l_2}| \\ &\leq \max_{m_1,m_2 \in \{1,\ldots,K_1\}} |\Delta_{I,m_1,m_2}| + \max_{m_1,m_2 \in \{1,\ldots,K_1\}} |N_{I,m_1} - N_{I,m_2}| \end{aligned}$$

so

$$\max_{m_1,m_2 \in \{1,\ldots,K_1\}} |\Delta_{I,m_1,m_2} + (N_{I,m_1} - N_{I,m_2})| \leq \max_{m_1,m_2 \in \{1,\ldots,K_1\}} |\Delta_{I,m_1,m_2}| + \max_{m_1,m_2 \in \{1,\ldots,K_1\}} |N_{I,m_1} - N_{I,m_2}| \tag{71}$$

Also, for all $l_1, l_2 \in \{1,\ldots,K_1\}$,

$$\begin{aligned} |\Delta_{I,l_1,l_2} + (N_{I,l_1} - N_{I,l_2})| &\geq |N_{I,l_1} - N_{I,l_2}| - |\Delta_{I,l_1,l_2}| \\ &\geq |N_{I,l_1} - N_{I,l_2}| - \max_{m_1,m_2 \in \{1,\ldots,K_1\}} |\Delta_{I,m_1,m_2}| \end{aligned}$$

so for all $l_1, l_2 \in \{1,\ldots,K_1\}$,

$$\begin{aligned} \max_{m_1,m_2 \in \{1,\ldots,K_1\}} |\Delta_{I,m_1,m_2} + (N_{I,m_1} - N_{I,m_2})| &\geq |\Delta_{I,l_1,l_2} + (N_{I,l_1} - N_{I,l_2})| \\ &\geq |N_{I,l_1} - N_{I,l_2}| - \max_{m_1,m_2 \in \{1,\ldots,K_1\}} |\Delta_{I,m_1,m_2}| \end{aligned}$$

and thus

$$\max_{m_1,m_2 \in \{1,\ldots,K_1\}} |\Delta_{I,m_1,m_2} + (N_{I,m_1} - N_{I,m_2})| \geq \max_{m_1,m_2 \in \{1,\ldots,K_1\}} |N_{I,m_1} - N_{I,m_2}| - \max_{m_1,m_2 \in \{1,\ldots,K_1\}} |\Delta_{I,m_1,m_2}| \tag{72}$$

Results (71) and (72) yield

$$A_I - B_I \leq C_I \leq A_I + B_I$$

where

$$A_I \equiv \max_{m_1,m_2 \in \{1,\ldots,K_1\}} |N_{I,m_1} - N_{I,m_2}|$$

$$B_I \equiv \max_{m_1,m_2 \in \{1,\ldots,K_1\}} |\Delta_{I,m_1,m_2}|$$

and

$$C_I \equiv \max_{m_1,m_2 \in \{1,\ldots,K_1\}} |\Delta_{I,m_1,m_2} + (N_{I,m_1} - N_{I,m_2})|$$

Thus, for any x,

$$\text{Prob}(A_I + B_I \leq x) \leq \text{Prob}(C_I \leq x) \leq \text{Prob}(A_I - B_I \leq x) \tag{73}$$

Now let $F_{\text{R}(\text{K}_1)}$ denote the distribution of the range of K_1 independent N(0,1)'s (that is, the distribution of A_I), and let an arbitrary $\epsilon > 0$ and an arbitrary x be given. Then, since $F_{\text{R}(\text{K}_1)}$ is a continuous function (see, for example, David (1981)), we can find a $\delta > 0$ such that

$$F_{\text{R}(\text{K}_1)}(x - \delta) > F_{\text{R}(\text{K}_1)}(x) - \epsilon \tag{74}$$

and

$$F_{\text{R}(\text{K}_1)}(x + \delta) < F_{\text{R}(\text{K}_1)}(x) + \epsilon \tag{75}$$

Also, by result (70), we can find an $N_{\epsilon,\delta}$ such that $I > N_{\epsilon,\delta}$ implies that

$$\text{Prob}(B_I \le \delta) > 1 - \epsilon \tag{76}$$

Thus, by results (74) and (76), (recall that A_I and B_I are statistically independent), for $I > N_{\epsilon,\delta}$,

$$\begin{aligned}\text{Prob}(A_I + B_I \le x) &\ge \text{Prob}(A_I \le x - \delta) \times \text{Prob}(B_I \le \delta) \\ &= F_{\text{R}(\text{K}_1)}(x - \delta) \times \text{Prob}(B_I \le \delta) \\ &> (F_{\text{R}(\text{K}_1)}(x) - \epsilon)(1 - \epsilon) > F_{\text{R}(\text{K}_1)}(x) - 2\epsilon \end{aligned} \tag{77}$$

Similarly,

$$\{A_I \le x + B_I\} \quad \subset \quad \{A_I \le x + B_I \text{ and } B_I \ge \delta\} \cup \{A_I \le x + B_I \text{ and } B_I < \delta\}$$

so

$$\text{Prob}(A_I \le x + B_I) \le \text{Prob}(B_I \ge \delta) + \text{Prob}(A_I \le x + \delta)$$

and for $I > N_{\epsilon,\delta}$, by results (75) and (76),

$$\text{Prob}(A_I - B_I \le x) < \epsilon + \text{Prob}(A_I \le x + \delta) < F_{\text{R}(\text{K}_1)}(x) + 2\epsilon \tag{78}$$

Results (69), (73), (77), and (78) imply that under a predictor sort allocation, for all x, as $I \to \infty$,

$$\text{Prob}\left(Q_I / \left(\sigma_Y \sqrt{1 - \rho^2}\right) \le x\right) \to F_{\text{R}(\text{K}_1)}(x)$$

or

$$Q_I / \left(\sigma_Y \sqrt{1 - \rho^2}\right) \xrightarrow{D} F_{\text{R}(\text{K}_1)} \tag{79}$$

As noted in results (32) and (34)

$$s_{\text{ub}} \xrightarrow{\text{p}} \sigma_Y \tag{80}$$

in the unblocked case, and

$$s_{\text{b}} \xrightarrow{\text{p}} \sigma_Y \sqrt{1 - \rho^2} \tag{81}$$

in the blocked case.

From results (79)–(81), the theorem follows.

14 Appendix G — Proof of Theorem 1 in the multi-factor case

14.1 Numerator of the test statistic

We assume that we have F factors, K_j levels in the jth factor, and I replicates per cell (or I blocks in the blocked case). For a test of the significance of the first factor, the numerator sum of squares in both the blocked and unblocked cases is

$$\text{SS}_{\text{num}} = \sum_{j_1=1}^{K_1} I K_2 \dots K_F (\bar{y}_{\cdot j_1 \cdot \dots \cdot} - \bar{y}_{\cdot \dots \cdot})^2$$

Thus, in the null case, using the notation introduced at the beginning of Section 5, we have

$$\begin{aligned}
\mathrm{SS}_{\mathrm{num}} &= \sum_{j_1=1}^{K_1} IK_2 \ldots K_F \times \sigma_Y^2 \left(\rho(\bar{x}_{k(\cdot j_1 \cdot \ldots \cdot),n} - \bar{x}_{k(\cdot \ldots \cdot),n}) + \sqrt{1-\rho^2}(\bar{p}_{\cdot j_1 \cdot \ldots \cdot} - \bar{p}_{\cdot \ldots \cdot})\right)^2 \\
&= \sum_{j_1=1}^{K_1} IK_2 \ldots K_F \times \sigma_Y^2 \times \rho^2 \left(\bar{x}_{k(\cdot j_1 \cdot \ldots \cdot),n} - \bar{x}_{k(\cdot \ldots \cdot),n}\right)^2 \\
&+ 2\sum_{j_1=1}^{K_1} IK_2 \ldots K_F \times \sigma_Y^2 \times \rho \times \sqrt{1-\rho^2} \left(\bar{x}_{k(\cdot j_1 \cdot \ldots \cdot),n} - \bar{x}_{k(\cdot \ldots \cdot),n}\right) (\bar{p}_{\cdot j_1 \cdot \ldots \cdot} - \bar{p}_{\cdot \ldots \cdot}) \\
&+ \sum_{j_1=1}^{K_1} IK_2 \ldots K_F \times \sigma_Y^2 \times (1-\rho^2) (\bar{p}_{\cdot j_1 \cdot \ldots \cdot} - \bar{p}_{\cdot \ldots \cdot})^2 \\
&\equiv T_1 + T_2 + T_3 \qquad (82)
\end{aligned}$$

It is well known that

$$T_3 \sim \sigma_Y^2 \times (1-\rho^2) \times \chi^2_{K_1-1} \qquad (83)$$

Also, for $j_1 \in \{1, \ldots, K_1\}$,

$$\begin{aligned}
\sqrt{I}\left|\bar{x}_{k(\cdot j_1 \cdot \ldots \cdot),n} - \bar{x}_{k(\cdot \ldots \cdot),n}\right| &= \sqrt{I}\left|\sum_{i=1}^{I} \left(\bar{x}_{k(ij_1 \cdot \ldots \cdot),n} - \bar{x}_{k(i \cdot \ldots \cdot),n}\right)/I\right| \\
&\le \sum_{i=1}^{I} \left|\bar{x}_{k(ij_1 \cdot \ldots \cdot),n} - \bar{x}_{k(i \cdot \ldots \cdot),n}\right|/\sqrt{I} \\
&\le \sum_{i=1}^{I} (\max x \text{ in } i\text{th block} - \min x \text{ in } i\text{th block})/\sqrt{I} \\
&\le (\max \text{ overall } x - \min \text{ overall } x)/\sqrt{I} \xrightarrow{p} 0
\end{aligned}$$

as $I \xrightarrow{p} \infty$ by the lemma in the appendix of Verrill (1993). Thus,

$$T_1 \xrightarrow{p} 0 \qquad (84)$$

as $I \xrightarrow{p} \infty$. By results (83) and (84) and the Cauchy-Schwarz theorem,

$$T_2 \xrightarrow{p} 0 \qquad (85)$$

Finally, by results (82), (83), (84), and (85), we have

$$\mathrm{SS}_{\mathrm{num}} \xrightarrow{D} \sigma_Y^2 \times (1-\rho^2) \times \chi^2_{K_1-1} \qquad (86)$$

14.2 Denominator of the test statistic in the predictor sort blocked case

We assume that we have F factors, K_j levels in the jth factor, and I blocks (formed by specimens with adjacent [randomized within a block] order statistics of the predictor). In this case the denominator sum of squares is

$$\mathrm{SS}_{\mathrm{den,bl}} = \sum_{i=1}^{I}\sum_{j_1=1}^{K_1} \cdots \sum_{j_F=1}^{K_F} (y_{ij_1 \ldots j_F} - (\bar{y}_{\cdot \ldots \cdot} + (\bar{y}_{i \cdot \ldots \cdot} - \bar{y}_{\cdot \ldots \cdot}) + \ldots + (\bar{y}_{\cdot \ldots \cdot j_F} - \bar{y}_{\cdot \ldots \cdot})))^2 \qquad (87)$$

It is easy to show that the μ's contained in equation (87) cancel and we are left with

$$\text{SS}_{\text{den,bl}} = \sum_{i=1}^{I} \sum_{j_1=1}^{K_1} \dots \sum_{j_F=1}^{K_F} (\text{diff}_{ij_1 \dots j_F})^2 \tag{88}$$

where, using notation similar to that introduced at the beginning of Section 5,

$$\begin{aligned} \text{diff}_{ij_1\dots j_F} &= w_{ij_1\dots j_F} - \bar{w}_{i\cdot\dots\cdot} \\ &- (\bar{w}_{\cdot j_1\cdot\dots\cdot} - \bar{w}_{\cdot\dots\cdot}) \\ &\vdots \\ &- (\bar{w}_{\cdot\dots\cdot j_F} - \bar{w}_{\cdot\dots\cdot}) \end{aligned}$$

and

$$w_{ij_1\dots j_F} = \sigma_Y \left(\rho x_{k(ij_1\dots j_F),n} + \sqrt{1-\rho^2} p_{ij_1\dots j_F} \right)$$

Now the F-statistic (nominal F) denominator in the blocked predictor sort case is

$$\text{SS}_{\text{den,bl}} / (IK_1 \dots K_F - (I + K_1 - 1 + \dots + K_F - 1))$$

We first claim that

$$\sum_{i=1}^{I} \sum_{j_1=1}^{K_1} \dots \sum_{j_F=1}^{K_F} (\bar{w}_{\cdot j_1\cdot\dots\cdot} - \bar{w}_{\cdot\dots\cdot})^2 / (IK_1 \dots K_F - (I + K_1 - 1 + \dots + K_F - 1)) \xrightarrow{p} 0 \tag{89}$$

(and the similar results for the corresponding $j_2, \dots, j_F$ terms). It is clear that (89) will follow if we can show that

$$\sum_{j_1=1}^{K_1} (\bar{w}_{\cdot j_1\cdot\dots\cdot} - \bar{w}_{\cdot\dots\cdot})^2 \xrightarrow{p} 0$$

as $I \to \infty$ which, in turn, will follow if

$$\sum_{j_1=1}^{K_1} |\bar{w}_{\cdot j_1\cdot\dots\cdot} - \bar{w}_{\cdot\dots\cdot}| \xrightarrow{p} 0 \tag{90}$$

as $I \to \infty$. We have

$$\bar{w}_{\cdot j_1\cdot\dots\cdot} - \bar{w}_{\cdot\dots\cdot} = \sigma_Y \left(\rho(\bar{x}_{k(\cdot j_1\cdot\dots\cdot),n} - \bar{x}_{k(\cdot\dots\cdot),n}) + \sqrt{1-\rho^2}(\bar{p}_{\cdot j_1\cdot\dots\cdot} - \bar{p}_{\cdot\dots\cdot}) \right) \tag{91}$$

We know that

$$\bar{p}_{\cdot j_1\cdot\dots\cdot} - \bar{p}_{\cdot\dots\cdot} \xrightarrow{p} 0 \tag{92}$$

as $I \to \infty$ because both $\bar{p}$'s are averages of more than I iid $\text{N}(0,1)$'s. We also have

$$\begin{aligned} \left|\bar{x}_{k(\cdot j_1\cdot\dots\cdot),n} - \bar{x}_{k(\cdot\dots\cdot),n}\right| &\leq \sum_{i=1}^{I} \left|\bar{x}_{k(ij_1\cdot\dots\cdot),n} - \bar{x}_{k(i\cdot\dots\cdot),n}\right| / I \\ &\leq \sum_{i=1}^{I} (\text{max } x \text{ in block } i - \text{min } x \text{ in block } i)/I \\ &\leq (\text{overall max } x - \text{overall min } x)/I \end{aligned} \tag{93}$$

which converges in probability to 0 as $I \to \infty$ by the lemma in the appendix of Verrill (1993). By results (91), (92), and (93), results (90) and thus (89) follow.

Next we claim that

$$\sum_{i=1}^{I}\sum_{j_1=1}^{K_1}\cdots\sum_{j_F=1}^{K_F}\left(w_{ij_1\ldots j_F}-\bar{w}_{i\cdot\ldots\cdot}\right)^2/(IK_1\ldots K_F-(I+K_1-1+\ldots+K_F-1)) \xrightarrow{p} (1-\rho^2)\sigma_Y^2 \quad (94)$$

We have

$$\begin{aligned}
\sum_{i=1}^{I}\sum_{j_1=1}^{K_1}\cdots\sum_{j_F=1}^{K_F}\left(w_{ij_1\ldots j_F}-\bar{w}_{i\cdot\ldots\cdot}\right)^2 &= \sum_{i=1}^{I}\sum_{j_1=1}^{K_1}\cdots\sum_{j_F=1}^{K_F}\rho^2\sigma_Y^2\left(x_{k(ij_1\ldots j_F),n}-\bar{x}_{k(i\cdot\ldots\cdot),n}\right)^2 \\
&+ \sum_{i=1}^{I}\sum_{j_1=1}^{K_1}\cdots\sum_{j_F=1}^{K_F}2\rho\sqrt{1-\rho^2}\sigma_Y^2\left(x_{k(ij_1\ldots j_F),n}-\bar{x}_{k(i\cdot\ldots\cdot),n}\right)\left(p_{ij_1\ldots j_F}-\bar{p}_{i\cdot\ldots\cdot}\right) \\
&+ \sum_{i=1}^{I}\sum_{j_1=1}^{K_1}\cdots\sum_{j_F=1}^{K_F}(1-\rho^2)\sigma_Y^2\left(p_{ij_1\ldots j_F}-\bar{p}_{i\cdot\ldots\cdot}\right)^2 \\
&\equiv T_1+T_2+T_3 \qquad (95)
\end{aligned}$$

First, we claim that

$$T_1/(IK_1\ldots K_F-(I+K_1-1+\ldots+K_F-1)) \xrightarrow{p} 0 \quad (96)$$

as $I \to \infty$. This follows from a slightly modified version of result (93) above:

$$\begin{aligned}
\sum_{i=1}^{I}\left|x_{k(ij_1\ldots j_F),n}-\bar{x}_{k(i\cdot\ldots\cdot),n}\right|/\sqrt{I} &\le \sum_{i=1}^{I}(\text{max } x \text{ in block } i - \text{min } x \text{ in block } i)/\sqrt{I} \\
&\le (\text{overall max } x - \text{overall min } x)/\sqrt{I}
\end{aligned}$$

which converges in probability to 0 as $I \to \infty$ by the lemma in the appendix of Verrill (1993), so

$$\sum_{i=1}^{I}\left(x_{k(ij_1\ldots j_F),n}-\bar{x}_{k(i\cdot\ldots\cdot),n}\right)^2/I \xrightarrow{p} 0$$

as $I \to \infty$.

Next, we claim that

$$T_3/(IK_1\ldots K_F-(I+K_1-1+\ldots+K_F-1)) \xrightarrow{p} (1-\rho^2)\sigma_Y^2 \quad (97)$$

We know that for $i \in \{1,\ldots,I\}$,

$$\sum_{j_1=1}^{K_1}\cdots\sum_{j_F=1}^{K_F}\left(p_{ij_1\ldots j_F}-\bar{p}_{i\cdot\ldots\cdot}\right)^2/(K_1\ldots K_F-1) \sim \chi^2_{K_1\ldots K_F-1}/(K_1\ldots K_F-1)$$

which has expectation 1. Thus,

$$\sum_{i=1}^{I}\left(\sum_{j_1=1}^{K_1}\cdots\sum_{j_F=1}^{K_F}\left(p_{ij_1\ldots j_F}-\bar{p}_{i\cdot\ldots\cdot}\right)^2/(K_1\ldots K_F-1)\right)/I \xrightarrow{p} 1$$

as $I \to \infty$ which establishes result (97).

Results (96) and (97) and the Cauchy-Schwarz theorem yield

$$T_2/(IK_1 \dots K_F - (I + K_1 - 1 + \dots + K_F - 1)) \xrightarrow{p} 0 \tag{98}$$

Results (95), (96), (97), and (98), yield result (94).

Results (88), (89), and (94), and the Cauchy-Schwarz theorem yield

$$\mathrm{SS}_{\mathrm{den,bl}}/(IK_1 \dots K_F - (I + K_1 - 1 + \dots + K_F - 1)) \xrightarrow{p} (1-\rho^2)\sigma_Y^2 \tag{99}$$

14.3 Denominator of the test statistic in the predictor sort unblocked case

We assume that we have F factors, K_j levels in the jth factor, and I blocks (formed by specimens with adjacent [randomized within a block] order statistics of the predictor). However, we further assume that despite such an allocation, the analysis ignores the "blocking" and acts as if there were I replicates for each cell. In this case the denominator sum of squares is

$$\mathrm{SS}_{\mathrm{den,unbl}} = \sum_{i=1}^{I} \sum_{j_1=1}^{K_1} \dots \sum_{j_F=1}^{K_F} (y_{ij_1\dots j_F} - (\bar{y}_{\cdot\dots\cdot} + (\bar{y}_{\cdot j_1 \cdot\dots\cdot} - \bar{y}_{\cdot\dots\cdot}) + \dots + (\bar{y}_{\cdot\dots\cdot j_F} - \bar{y}_{\cdot\dots\cdot})))^2 \tag{100}$$

It is easy to show that the μ's contained in equation (100) cancel and we are left with

$$\mathrm{SS}_{\mathrm{den,unbl}} = \sum_{i=1}^{I} \sum_{j_1=1}^{K_1} \dots \sum_{j_F=1}^{K_F} (\mathrm{diff}_{ij_1\dots j_F})^2 \tag{101}$$

where, using notation similar to that introduced at the beginning of Section 5,

$$\begin{aligned} \mathrm{diff}_{ij_1\dots j_F} &= w_{ij_1\dots j_F} - \bar{w}_{\cdot\dots\cdot} \\ &- (\bar{w}_{\cdot j_1 \cdot\dots\cdot} - \bar{w}_{\cdot\dots\cdot}) \\ &\vdots \\ &- (\bar{w}_{\cdot\dots\cdot j_F} - \bar{w}_{\cdot\dots\cdot}) \end{aligned}$$

and

$$w_{ij_1\dots j_F} = \sigma_Y \left(\rho x_{k(ij_1\dots j_F),n} + \sqrt{1-\rho^2} z_{ij_1\dots j_F} \right)$$

Now the F-statistic (nominal F) denominator in the unblocked predictor sort case is

$$\mathrm{SS}_{\mathrm{den,unbl}}/(IK_1 \dots K_F - (K_1 + K_2 - 1 + \dots + K_F - 1))$$

We first claim that

$$\sum_{i=1}^{I} \sum_{j_1=1}^{K_1} \dots \sum_{j_F=1}^{K_F} (\bar{w}_{\cdot j_1 \cdot\dots\cdot} - \bar{w}_{\cdot\dots\cdot})^2 / (IK_1 \dots K_F - (K_1 + K_2 - 1 + \dots + K_F - 1)) \xrightarrow{p} 0 \tag{102}$$

(and the similar results for the corresponding $j_2, \dots, j_F$ terms). This follows from the reasoning used to establish result (89).

Next we claim that

$$\sum_{i=1}^{I} \sum_{j_1=1}^{K_1} \dots \sum_{j_F=1}^{K_F} (w_{ij_1\dots j_F} - \bar{w}_{\cdot\dots\cdot})^2 / (IK_1 \dots K_F - (K_1 + K_2 - 1 + \dots + K_F - 1)) \xrightarrow{p} \sigma_Y^2 \tag{103}$$

This follows immediately from the fact that the summation in (103) is just a reordering of the summation

$$\sum_{i=1}^{I}\sum_{j_1=1}^{K_1}\cdots\sum_{j_F=1}^{K_F}(s_{ij_1\ldots j_F}-\bar{s}_{\cdot\ldots\cdot})^2$$

where the s's are iid $\mathrm{N}(0,\sigma_Y^2)$'s.

Results (101), (102), and (103), and the Cauchy-Schwarz theorem yield

$$\mathrm{SS}_{\mathrm{den,unbl}}/(IK_1\ldots K_F-(K_1+K_2-1+\ldots+K_F-1))\xrightarrow{p}\sigma_Y^2 \tag{104}$$

ρ	J	I	m	no ps alloc 1-way anal	ps alloc 1-way analysis no ρ	$\hat{\rho}$	true ρ	ps alloc 2-way anal	ancovar no ps alloc	ps alloc	theoretical power 1-way	2-way
0.50	2	3	1	0.052	0.034	0.124	0.051	0.050	0.052	0.051	0.050	0.050
0.50	2	3	2	0.051	0.034	0.126	0.052	0.050	0.052	0.051	0.050	0.050
0.50	2	3	3	0.052	0.035	0.127	0.053	0.051	0.053	0.052	0.051	0.051
0.50	2	3	4	0.053	0.035	0.130	0.054	0.052	0.054	0.053	0.053	0.052
0.50	2	3	5	0.055	0.037	0.133	0.057	0.054	0.055	0.055	0.056	0.054
0.50	2	3	6	0.058	0.040	0.138	0.060	0.056	0.057	0.058	0.059	0.056
0.50	2	3	7	0.060	0.043	0.143	0.063	0.058	0.060	0.061	0.063	0.058
0.50	2	3	8	0.064	0.045	0.150	0.068	0.060	0.063	0.064	0.067	0.061
0.50	2	3	9	0.068	0.049	0.157	0.073	0.063	0.067	0.069	0.073	0.065
0.50	2	3	10	0.073	0.052	0.165	0.078	0.066	0.072	0.074	0.079	0.069
0.50	2	3	11	0.078	0.057	0.175	0.085	0.070	0.076	0.079	0.086	0.073
0.50	2	3	12	0.084	0.063	0.184	0.093	0.074	0.082	0.085	0.093	0.078
0.50	2	3	13	0.090	0.068	0.195	0.102	0.079	0.088	0.092	0.102	0.083
0.50	2	3	14	0.097	0.075	0.207	0.110	0.085	0.094	0.100	0.111	0.088
0.50	2	3	15	0.103	0.082	0.220	0.120	0.090	0.100	0.108	0.121	0.094
0.50	2	3	16	0.111	0.089	0.233	0.129	0.096	0.107	0.116	0.131	0.101
0.50	2	3	17	0.120	0.097	0.248	0.139	0.102	0.116	0.124	0.143	0.107
0.50	2	3	18	0.129	0.106	0.264	0.151	0.109	0.124	0.134	0.155	0.115
0.50	2	3	19	0.140	0.116	0.279	0.164	0.115	0.133	0.145	0.168	0.122
0.50	2	3	20	0.149	0.125	0.294	0.177	0.123	0.141	0.155	0.181	0.130
0.50	2	3	21	0.160	0.135	0.310	0.190	0.131	0.151	0.165	0.195	0.138
0.50	2	5	1	0.050	0.030	0.074	0.051	0.048	0.050	0.049	0.050	0.050
0.50	2	5	2	0.051	0.030	0.076	0.051	0.049	0.050	0.050	0.051	0.051
0.50	2	5	3	0.052	0.031	0.078	0.054	0.051	0.052	0.053	0.053	0.052
0.50	2	5	4	0.054	0.033	0.082	0.058	0.055	0.055	0.057	0.057	0.055
0.50	2	5	5	0.059	0.037	0.088	0.062	0.059	0.059	0.062	0.062	0.059
0.50	2	5	6	0.064	0.042	0.097	0.070	0.065	0.065	0.069	0.069	0.065
0.50	2	5	7	0.070	0.047	0.107	0.078	0.072	0.073	0.076	0.077	0.071
0.50	2	5	8	0.078	0.054	0.120	0.087	0.080	0.081	0.086	0.087	0.079
0.50	2	5	9	0.087	0.062	0.134	0.099	0.088	0.091	0.098	0.099	0.088
0.50	2	5	10	0.097	0.071	0.149	0.113	0.099	0.102	0.110	0.112	0.099
0.50	2	5	11	0.108	0.082	0.165	0.129	0.110	0.116	0.125	0.127	0.110
0.50	2	5	12	0.120	0.094	0.185	0.146	0.121	0.131	0.140	0.144	0.123
0.50	2	5	13	0.134	0.107	0.207	0.163	0.135	0.146	0.158	0.162	0.137
0.50	2	5	14	0.148	0.122	0.230	0.182	0.150	0.163	0.178	0.182	0.152
0.50	2	5	15	0.163	0.138	0.252	0.204	0.166	0.181	0.198	0.204	0.169
0.50	2	5	16	0.180	0.155	0.278	0.227	0.182	0.201	0.219	0.227	0.186
0.50	2	5	17	0.198	0.173	0.304	0.253	0.200	0.222	0.243	0.252	0.205
0.50	2	5	18	0.218	0.194	0.332	0.277	0.219	0.243	0.268	0.278	0.225
0.50	2	5	19	0.238	0.217	0.359	0.305	0.239	0.266	0.294	0.305	0.246
0.50	2	5	20	0.260	0.241	0.389	0.332	0.260	0.291	0.322	0.333	0.268
0.50	2	5	21	0.283	0.264	0.420	0.361	0.283	0.316	0.352	0.363	0.290

Table 1: Power simulation (see Section 2 for details), $\rho = 0.5$, $J = 2$, $I = 3, 5$

				no ps alloc	ps alloc 1-way analysis			ps alloc	ancovar		theoretical power	
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.50	2	10	1	0.050	0.026	0.059	0.051	0.048	0.052	0.050	0.050	0.050
0.50	2	10	2	0.052	0.028	0.061	0.051	0.051	0.053	0.052	0.052	0.052
0.50	2	10	3	0.056	0.031	0.066	0.056	0.055	0.058	0.057	0.057	0.056
0.50	2	10	4	0.062	0.036	0.076	0.065	0.062	0.065	0.065	0.066	0.064
0.50	2	10	5	0.072	0.044	0.089	0.077	0.073	0.076	0.078	0.078	0.075
0.50	2	10	6	0.081	0.054	0.106	0.092	0.089	0.090	0.094	0.094	0.089
0.50	2	10	7	0.096	0.067	0.127	0.111	0.107	0.110	0.113	0.114	0.107
0.50	2	10	8	0.114	0.083	0.152	0.136	0.127	0.133	0.136	0.137	0.128
0.50	2	10	9	0.134	0.101	0.181	0.163	0.153	0.158	0.163	0.165	0.153
0.50	2	10	10	0.157	0.125	0.215	0.195	0.180	0.188	0.196	0.196	0.180
0.50	2	10	11	0.183	0.152	0.251	0.230	0.210	0.219	0.230	0.231	0.212
0.50	2	10	12	0.212	0.182	0.290	0.268	0.243	0.254	0.268	0.270	0.246
0.50	2	10	13	0.245	0.216	0.333	0.309	0.280	0.294	0.309	0.311	0.284
0.50	2	10	14	0.278	0.252	0.377	0.352	0.319	0.334	0.353	0.356	0.323
0.50	2	10	15	0.315	0.291	0.424	0.398	0.360	0.378	0.398	0.402	0.366
0.50	2	10	16	0.354	0.332	0.472	0.446	0.403	0.425	0.446	0.450	0.409
0.50	2	10	17	0.393	0.378	0.521	0.495	0.449	0.471	0.495	0.498	0.454
0.50	2	10	18	0.433	0.424	0.568	0.542	0.493	0.519	0.544	0.547	0.500
0.50	2	10	19	0.475	0.471	0.617	0.590	0.538	0.566	0.592	0.594	0.545
0.50	2	10	20	0.518	0.518	0.664	0.638	0.583	0.611	0.639	0.641	0.590
0.50	2	10	21	0.561	0.566	0.706	0.683	0.630	0.657	0.684	0.685	0.634
0.50	2	20	1	0.050	0.025	0.054	0.051	0.050	0.051	0.050	0.050	0.050
0.50	2	20	2	0.054	0.027	0.056	0.054	0.053	0.054	0.054	0.054	0.053
0.50	2	20	3	0.062	0.033	0.068	0.064	0.062	0.065	0.065	0.065	0.064
0.50	2	20	4	0.077	0.045	0.087	0.082	0.080	0.084	0.082	0.083	0.082
0.50	2	20	5	0.098	0.061	0.113	0.107	0.104	0.110	0.107	0.110	0.107
0.50	2	20	6	0.123	0.085	0.149	0.140	0.137	0.143	0.142	0.144	0.140
0.50	2	20	7	0.156	0.115	0.194	0.185	0.179	0.184	0.186	0.187	0.180
0.50	2	20	8	0.194	0.152	0.246	0.236	0.225	0.233	0.236	0.238	0.229
0.50	2	20	9	0.238	0.201	0.306	0.295	0.279	0.289	0.296	0.296	0.284
0.50	2	20	10	0.287	0.254	0.370	0.359	0.343	0.352	0.361	0.360	0.345
0.50	2	20	11	0.344	0.314	0.442	0.429	0.411	0.418	0.431	0.429	0.411
0.50	2	20	12	0.402	0.380	0.512	0.499	0.480	0.489	0.502	0.499	0.479
0.50	2	20	13	0.462	0.449	0.582	0.570	0.550	0.560	0.571	0.570	0.548
0.50	2	20	14	0.522	0.519	0.651	0.637	0.619	0.629	0.640	0.638	0.615
0.50	2	20	15	0.582	0.590	0.714	0.702	0.681	0.694	0.705	0.702	0.679
0.50	2	20	16	0.640	0.655	0.769	0.761	0.738	0.753	0.763	0.761	0.738
0.50	2	20	17	0.697	0.718	0.821	0.813	0.791	0.804	0.814	0.812	0.791
0.50	2	20	18	0.748	0.775	0.863	0.857	0.837	0.850	0.858	0.856	0.837
0.50	2	20	19	0.794	0.825	0.899	0.893	0.875	0.888	0.894	0.893	0.876
0.50	2	20	20	0.834	0.867	0.927	0.922	0.907	0.917	0.923	0.922	0.908
0.50	2	20	21	0.870	0.900	0.948	0.945	0.934	0.941	0.945	0.945	0.933

Table 1: Power simulation (see Section 2 for details), $\rho = 0.5$, $J = 2$, $I = 10, 20$

				no ps alloc	ps alloc			ps alloc	ancovar		theoretical power	
					1-way analysis							
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.50	2	40	1	0.050	0.025	0.053	0.051	0.051	0.050	0.051	0.050	0.050
0.50	2	40	2	0.055	0.030	0.060	0.058	0.058	0.057	0.058	0.057	0.057
0.50	2	40	3	0.073	0.044	0.083	0.080	0.080	0.080	0.080	0.080	0.080
0.50	2	40	4	0.102	0.069	0.123	0.120	0.118	0.119	0.121	0.119	0.118
0.50	2	40	5	0.145	0.109	0.178	0.175	0.171	0.176	0.175	0.175	0.172
0.50	2	40	6	0.199	0.163	0.253	0.248	0.243	0.248	0.248	0.247	0.242
0.50	2	40	7	0.263	0.234	0.340	0.334	0.329	0.333	0.336	0.334	0.327
0.50	2	40	8	0.340	0.317	0.437	0.432	0.423	0.429	0.432	0.431	0.422
0.50	2	40	9	0.423	0.413	0.538	0.534	0.523	0.528	0.534	0.532	0.522
0.50	2	40	10	0.509	0.513	0.636	0.632	0.620	0.628	0.633	0.631	0.620
0.50	2	40	11	0.599	0.614	0.727	0.721	0.710	0.719	0.723	0.722	0.712
0.50	2	40	12	0.682	0.705	0.805	0.801	0.790	0.795	0.801	0.801	0.791
0.50	2	40	13	0.756	0.786	0.868	0.863	0.855	0.859	0.865	0.864	0.856
0.50	2	40	14	0.819	0.852	0.914	0.911	0.904	0.911	0.912	0.912	0.905
0.50	2	40	15	0.870	0.902	0.946	0.945	0.940	0.945	0.945	0.946	0.941
0.50	2	40	16	0.913	0.939	0.968	0.967	0.963	0.969	0.968	0.969	0.965
0.50	2	40	17	0.944	0.963	0.983	0.982	0.980	0.983	0.983	0.983	0.981
0.50	2	40	18	0.964	0.980	0.991	0.991	0.989	0.991	0.991	0.991	0.990
0.50	2	40	19	0.978	0.989	0.995	0.995	0.995	0.996	0.995	0.996	0.995
0.50	2	40	20	0.988	0.995	0.998	0.998	0.997	0.998	0.998	0.998	0.998
0.50	2	40	21	0.994	0.997	0.999	0.999	0.999	0.999	0.999	0.999	0.999

Table 1: Power simulation (see Section 2 for details), $\rho = 0.5$, $J = 2$, $I = 40$

					ps alloc				ancovar		theoretical power	
				no ps alloc	1-way analysis			ps alloc				
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.50	5	3	1	0.049	0.020	0.092	0.046	0.049	0.051	0.047	0.050	0.050
0.50	5	3	2	0.049	0.020	0.092	0.046	0.049	0.052	0.048	0.050	0.050
0.50	5	3	3	0.049	0.020	0.092	0.047	0.049	0.052	0.049	0.051	0.051
0.50	5	3	4	0.050	0.020	0.092	0.047	0.050	0.053	0.050	0.051	0.051
0.50	5	3	5	0.051	0.021	0.094	0.049	0.051	0.054	0.051	0.053	0.052
0.50	5	3	6	0.052	0.022	0.096	0.050	0.052	0.055	0.053	0.054	0.054
0.50	5	3	7	0.053	0.023	0.098	0.052	0.054	0.057	0.054	0.056	0.055
0.50	5	3	8	0.055	0.023	0.100	0.054	0.055	0.059	0.056	0.058	0.057
0.50	5	3	9	0.057	0.025	0.103	0.056	0.057	0.061	0.059	0.061	0.060
0.50	5	3	10	0.059	0.026	0.107	0.059	0.060	0.064	0.062	0.064	0.062
0.50	5	3	11	0.062	0.028	0.111	0.062	0.062	0.067	0.065	0.067	0.065
0.50	5	3	12	0.065	0.029	0.115	0.065	0.065	0.070	0.068	0.071	0.069
0.50	5	3	13	0.068	0.031	0.119	0.069	0.068	0.074	0.072	0.075	0.073
0.50	5	3	14	0.072	0.034	0.125	0.073	0.072	0.078	0.076	0.079	0.077
0.50	5	3	15	0.076	0.036	0.131	0.077	0.076	0.083	0.081	0.084	0.081
0.50	5	3	16	0.080	0.039	0.138	0.082	0.081	0.088	0.086	0.090	0.086
0.50	5	3	17	0.084	0.042	0.145	0.088	0.086	0.094	0.092	0.096	0.092
0.50	5	3	18	0.089	0.045	0.153	0.095	0.091	0.099	0.098	0.102	0.097
0.50	5	3	19	0.094	0.049	0.161	0.101	0.097	0.105	0.104	0.109	0.104
0.50	5	3	20	0.099	0.053	0.169	0.108	0.104	0.111	0.111	0.117	0.111
0.50	5	3	21	0.105	0.057	0.178	0.115	0.111	0.119	0.119	0.125	0.118
0.50	5	5	1	0.047	0.018	0.066	0.048	0.051	0.049	0.050	0.050	0.050
0.50	5	5	2	0.047	0.018	0.066	0.049	0.051	0.049	0.050	0.050	0.050
0.50	5	5	3	0.048	0.019	0.068	0.050	0.052	0.051	0.051	0.051	0.051
0.50	5	5	4	0.049	0.019	0.070	0.052	0.053	0.052	0.053	0.053	0.053
0.50	5	5	5	0.051	0.020	0.073	0.054	0.055	0.054	0.056	0.055	0.055
0.50	5	5	6	0.054	0.021	0.076	0.057	0.058	0.058	0.059	0.058	0.058
0.50	5	5	7	0.057	0.023	0.080	0.060	0.061	0.061	0.063	0.062	0.062
0.50	5	5	8	0.060	0.025	0.086	0.064	0.065	0.066	0.068	0.067	0.066
0.50	5	5	9	0.065	0.028	0.092	0.069	0.070	0.071	0.072	0.072	0.071
0.50	5	5	10	0.069	0.030	0.099	0.075	0.076	0.078	0.079	0.078	0.077
0.50	5	5	11	0.074	0.033	0.108	0.083	0.084	0.085	0.086	0.086	0.084
0.50	5	5	12	0.080	0.037	0.117	0.090	0.091	0.092	0.095	0.094	0.092
0.50	5	5	13	0.086	0.041	0.127	0.098	0.099	0.100	0.104	0.103	0.100
0.50	5	5	14	0.094	0.046	0.139	0.109	0.109	0.110	0.115	0.113	0.110
0.50	5	5	15	0.102	0.052	0.152	0.119	0.120	0.122	0.126	0.125	0.121
0.50	5	5	16	0.111	0.058	0.165	0.133	0.129	0.133	0.138	0.137	0.132
0.50	5	5	17	0.121	0.066	0.180	0.146	0.142	0.146	0.152	0.151	0.145
0.50	5	5	18	0.131	0.073	0.195	0.160	0.155	0.160	0.167	0.166	0.159
0.50	5	5	19	0.143	0.083	0.212	0.174	0.170	0.175	0.183	0.183	0.175
0.50	5	5	20	0.155	0.093	0.230	0.191	0.185	0.192	0.200	0.200	0.191
0.50	5	5	21	0.169	0.104	0.249	0.210	0.203	0.209	0.218	0.219	0.209

Table 1: Power simulation (see Section 2 for details), $\rho = 0.5$, $J = 5$, $I = 3, 5$

				no ps alloc	ps alloc 1-way analysis			ps alloc	ancovar		theoretical power	
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.50	5	10	1	0.047	0.016	0.057	0.049	0.051	0.046	0.051	0.050	0.050
0.50	5	10	2	0.048	0.016	0.058	0.050	0.052	0.047	0.052	0.051	0.051
0.50	5	10	3	0.050	0.017	0.061	0.053	0.055	0.049	0.054	0.053	0.053
0.50	5	10	4	0.053	0.018	0.065	0.056	0.060	0.053	0.058	0.057	0.057
0.50	5	10	5	0.057	0.021	0.072	0.061	0.064	0.058	0.064	0.062	0.062
0.50	5	10	6	0.062	0.024	0.080	0.068	0.072	0.065	0.071	0.069	0.069
0.50	5	10	7	0.069	0.028	0.089	0.078	0.080	0.075	0.079	0.079	0.078
0.50	5	10	8	0.077	0.033	0.100	0.089	0.091	0.086	0.091	0.090	0.089
0.50	5	10	9	0.087	0.039	0.114	0.103	0.104	0.099	0.105	0.103	0.102
0.50	5	10	10	0.099	0.046	0.131	0.118	0.120	0.115	0.121	0.119	0.117
0.50	5	10	11	0.114	0.056	0.151	0.136	0.137	0.134	0.140	0.138	0.135
0.50	5	10	12	0.129	0.067	0.174	0.158	0.157	0.154	0.161	0.159	0.156
0.50	5	10	13	0.145	0.080	0.198	0.181	0.179	0.178	0.184	0.183	0.179
0.50	5	10	14	0.165	0.095	0.225	0.207	0.206	0.204	0.211	0.210	0.206
0.50	5	10	15	0.187	0.113	0.255	0.237	0.234	0.233	0.242	0.240	0.235
0.50	5	10	16	0.211	0.135	0.288	0.269	0.265	0.267	0.273	0.273	0.267
0.50	5	10	17	0.237	0.158	0.324	0.303	0.300	0.301	0.309	0.308	0.301
0.50	5	10	18	0.265	0.184	0.363	0.340	0.335	0.339	0.346	0.346	0.338
0.50	5	10	19	0.295	0.213	0.403	0.380	0.374	0.378	0.386	0.386	0.377
0.50	5	10	20	0.325	0.246	0.444	0.420	0.413	0.418	0.428	0.428	0.418
0.50	5	10	21	0.359	0.280	0.486	0.463	0.454	0.460	0.472	0.471	0.460
0.50	5	20	1	0.049	0.013	0.051	0.048	0.049	0.048	0.049	0.050	0.050
0.50	5	20	2	0.050	0.014	0.054	0.050	0.051	0.050	0.051	0.052	0.052
0.50	5	20	3	0.054	0.016	0.060	0.056	0.056	0.055	0.056	0.056	0.056
0.50	5	20	4	0.060	0.020	0.068	0.064	0.065	0.064	0.064	0.065	0.064
0.50	5	20	5	0.069	0.025	0.080	0.076	0.077	0.076	0.077	0.077	0.076
0.50	5	20	6	0.081	0.033	0.097	0.091	0.092	0.093	0.093	0.093	0.092
0.50	5	20	7	0.096	0.042	0.119	0.112	0.114	0.114	0.115	0.114	0.113
0.50	5	20	8	0.116	0.056	0.146	0.138	0.139	0.141	0.142	0.141	0.140
0.50	5	20	9	0.139	0.072	0.178	0.171	0.169	0.173	0.172	0.174	0.172
0.50	5	20	10	0.167	0.094	0.215	0.207	0.205	0.212	0.210	0.212	0.210
0.50	5	20	11	0.198	0.120	0.260	0.252	0.249	0.258	0.254	0.257	0.254
0.50	5	20	12	0.236	0.152	0.312	0.302	0.299	0.308	0.304	0.308	0.304
0.50	5	20	13	0.277	0.191	0.368	0.356	0.354	0.363	0.362	0.363	0.359
0.50	5	20	14	0.325	0.237	0.430	0.415	0.415	0.421	0.422	0.423	0.418
0.50	5	20	15	0.376	0.288	0.491	0.477	0.476	0.483	0.483	0.486	0.480
0.50	5	20	16	0.428	0.344	0.555	0.542	0.540	0.545	0.547	0.549	0.543
0.50	5	20	17	0.480	0.403	0.617	0.605	0.602	0.606	0.611	0.612	0.606
0.50	5	20	18	0.533	0.466	0.677	0.668	0.663	0.667	0.671	0.673	0.666
0.50	5	20	19	0.589	0.534	0.734	0.722	0.719	0.724	0.729	0.729	0.723
0.50	5	20	20	0.641	0.599	0.783	0.774	0.770	0.776	0.779	0.781	0.775
0.50	5	20	21	0.694	0.661	0.827	0.820	0.817	0.823	0.824	0.827	0.821

Table 1: Power simulation (see Section 2 for details), $\rho = 0.5$, $J = 5$, $I = 10, 20$

					ps alloc						theoretical	
				no ps alloc	1-way analysis			ps alloc	ancovar		power	
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.50	5	40	1	0.049	0.014	0.050	0.049	0.049	0.051	0.050	0.050	0.050
0.50	5	40	2	0.052	0.015	0.054	0.053	0.053	0.055	0.053	0.053	0.053
0.50	5	40	3	0.060	0.019	0.064	0.062	0.063	0.064	0.062	0.063	0.063
0.50	5	40	4	0.074	0.026	0.082	0.080	0.080	0.083	0.080	0.081	0.081
0.50	5	40	5	0.093	0.037	0.110	0.107	0.107	0.108	0.108	0.108	0.108
0.50	5	40	6	0.120	0.055	0.148	0.144	0.145	0.147	0.145	0.146	0.145
0.50	5	40	7	0.156	0.082	0.198	0.193	0.194	0.196	0.195	0.196	0.195
0.50	5	40	8	0.201	0.119	0.260	0.256	0.255	0.258	0.258	0.258	0.257
0.50	5	40	9	0.254	0.168	0.335	0.330	0.329	0.330	0.333	0.333	0.331
0.50	5	40	10	0.316	0.231	0.420	0.415	0.413	0.414	0.416	0.417	0.414
0.50	5	40	11	0.388	0.307	0.512	0.505	0.504	0.504	0.510	0.506	0.503
0.50	5	40	12	0.464	0.393	0.603	0.598	0.596	0.594	0.601	0.597	0.594
0.50	5	40	13	0.543	0.486	0.690	0.684	0.684	0.678	0.687	0.684	0.681
0.50	5	40	14	0.621	0.581	0.766	0.760	0.760	0.758	0.763	0.762	0.759
0.50	5	40	15	0.696	0.671	0.831	0.827	0.827	0.826	0.830	0.829	0.826
0.50	5	40	16	0.762	0.751	0.884	0.881	0.879	0.880	0.883	0.883	0.881
0.50	5	40	17	0.818	0.820	0.924	0.922	0.921	0.921	0.923	0.923	0.922
0.50	5	40	18	0.869	0.875	0.953	0.952	0.951	0.952	0.952	0.953	0.951
0.50	5	40	19	0.908	0.918	0.972	0.971	0.971	0.972	0.971	0.972	0.971
0.50	5	40	20	0.939	0.950	0.983	0.983	0.983	0.984	0.983	0.984	0.984
0.50	5	40	21	0.960	0.970	0.991	0.991	0.991	0.992	0.991	0.992	0.991

Table 1: Power simulation (see Section 2 for details), $\rho = 0.5$, $J = 5$, $I = 40$

					ps alloc						theoretical	
				no ps alloc	1-way analysis			ps alloc	ancovar		power	
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.70	2	3	1	0.050	0.019	0.152	0.056	0.049	0.050	0.050	0.050	0.050
0.70	2	3	2	0.051	0.019	0.154	0.056	0.049	0.050	0.050	0.051	0.050
0.70	2	3	3	0.051	0.020	0.156	0.056	0.050	0.051	0.052	0.052	0.051
0.70	2	3	4	0.053	0.021	0.160	0.059	0.051	0.053	0.053	0.055	0.053
0.70	2	3	5	0.055	0.022	0.165	0.064	0.053	0.056	0.056	0.058	0.055
0.70	2	3	6	0.058	0.024	0.171	0.068	0.056	0.059	0.060	0.063	0.058
0.70	2	3	7	0.061	0.026	0.178	0.073	0.058	0.063	0.065	0.069	0.062
0.70	2	3	8	0.064	0.029	0.187	0.080	0.062	0.067	0.070	0.076	0.067
0.70	2	3	9	0.068	0.032	0.198	0.087	0.066	0.073	0.077	0.084	0.072
0.70	2	3	10	0.073	0.035	0.210	0.096	0.071	0.079	0.085	0.093	0.077
0.70	2	3	11	0.078	0.040	0.223	0.105	0.076	0.086	0.093	0.103	0.083
0.70	2	3	12	0.084	0.045	0.236	0.116	0.081	0.093	0.102	0.114	0.090
0.70	2	3	13	0.089	0.050	0.249	0.128	0.087	0.101	0.112	0.127	0.098
0.70	2	3	14	0.096	0.055	0.265	0.140	0.092	0.110	0.123	0.140	0.106
0.70	2	3	15	0.103	0.062	0.281	0.154	0.100	0.120	0.134	0.154	0.114
0.70	2	3	16	0.110	0.068	0.299	0.168	0.107	0.130	0.146	0.170	0.124
0.70	2	3	17	0.119	0.076	0.317	0.184	0.115	0.141	0.159	0.187	0.133
0.70	2	3	18	0.128	0.084	0.336	0.199	0.123	0.153	0.173	0.204	0.143
0.70	2	3	19	0.137	0.093	0.355	0.216	0.132	0.165	0.187	0.223	0.154
0.70	2	3	20	0.148	0.102	0.375	0.234	0.140	0.178	0.203	0.242	0.165
0.70	2	3	21	0.158	0.113	0.395	0.253	0.149	0.191	0.218	0.263	0.177
0.70	2	5	1	0.050	0.014	0.090	0.057	0.050	0.048	0.050	0.050	0.050
0.70	2	5	2	0.050	0.015	0.090	0.057	0.051	0.050	0.052	0.051	0.051
0.70	2	5	3	0.053	0.016	0.094	0.061	0.054	0.053	0.055	0.054	0.053
0.70	2	5	4	0.056	0.017	0.100	0.066	0.058	0.059	0.061	0.060	0.058
0.70	2	5	5	0.060	0.020	0.108	0.074	0.063	0.065	0.069	0.068	0.064
0.70	2	5	6	0.066	0.023	0.120	0.084	0.070	0.073	0.078	0.078	0.072
0.70	2	5	7	0.074	0.027	0.136	0.097	0.079	0.084	0.091	0.090	0.082
0.70	2	5	8	0.080	0.033	0.153	0.111	0.088	0.098	0.103	0.105	0.093
0.70	2	5	9	0.090	0.039	0.172	0.127	0.099	0.113	0.120	0.123	0.107
0.70	2	5	10	0.099	0.047	0.194	0.148	0.113	0.130	0.139	0.143	0.122
0.70	2	5	11	0.110	0.056	0.219	0.169	0.129	0.149	0.161	0.165	0.139
0.70	2	5	12	0.123	0.066	0.245	0.192	0.146	0.169	0.185	0.190	0.158
0.70	2	5	13	0.136	0.078	0.272	0.217	0.165	0.192	0.210	0.217	0.178
0.70	2	5	14	0.151	0.091	0.302	0.244	0.185	0.217	0.239	0.246	0.200
0.70	2	5	15	0.167	0.106	0.334	0.273	0.206	0.245	0.267	0.277	0.224
0.70	2	5	16	0.186	0.124	0.367	0.304	0.229	0.274	0.299	0.310	0.250
0.70	2	5	17	0.205	0.143	0.400	0.336	0.254	0.305	0.331	0.345	0.277
0.70	2	5	18	0.223	0.162	0.435	0.370	0.279	0.336	0.366	0.381	0.305
0.70	2	5	19	0.244	0.183	0.471	0.405	0.305	0.370	0.401	0.419	0.334
0.70	2	5	20	0.266	0.206	0.506	0.440	0.330	0.401	0.437	0.457	0.364
0.70	2	5	21	0.287	0.231	0.542	0.476	0.359	0.436	0.473	0.495	0.395

Table 1: Power simulation (see Section 2 for details), $\rho = 0.7$, $J = 2$, $I = 3, 5$

				no ps alloc	ps alloc 1-way analysis			ps alloc	ancovar		theoretical power	
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.70	2	10	1	0.050	0.009	0.063	0.053	0.051	0.051	0.050	0.050	0.050
0.70	2	10	2	0.051	0.010	0.066	0.055	0.052	0.053	0.052	0.053	0.052
0.70	2	10	3	0.056	0.012	0.076	0.063	0.057	0.060	0.058	0.060	0.059
0.70	2	10	4	0.062	0.016	0.088	0.074	0.068	0.070	0.071	0.073	0.071
0.70	2	10	5	0.071	0.021	0.107	0.090	0.083	0.087	0.088	0.091	0.087
0.70	2	10	6	0.083	0.029	0.131	0.114	0.102	0.109	0.112	0.115	0.108
0.70	2	10	7	0.098	0.039	0.161	0.141	0.127	0.136	0.140	0.144	0.134
0.70	2	10	8	0.116	0.052	0.197	0.177	0.156	0.169	0.174	0.180	0.166
0.70	2	10	9	0.136	0.068	0.241	0.216	0.190	0.208	0.216	0.220	0.202
0.70	2	10	10	0.159	0.087	0.288	0.262	0.230	0.253	0.263	0.266	0.243
0.70	2	10	11	0.184	0.110	0.339	0.309	0.272	0.301	0.313	0.317	0.288
0.70	2	10	12	0.213	0.140	0.393	0.360	0.320	0.351	0.366	0.371	0.338
0.70	2	10	13	0.245	0.172	0.449	0.416	0.369	0.407	0.421	0.428	0.390
0.70	2	10	14	0.278	0.211	0.506	0.474	0.420	0.463	0.481	0.487	0.444
0.70	2	10	15	0.314	0.253	0.563	0.531	0.473	0.518	0.539	0.545	0.499
0.70	2	10	16	0.353	0.299	0.618	0.589	0.527	0.576	0.595	0.603	0.554
0.70	2	10	17	0.393	0.349	0.673	0.646	0.581	0.631	0.651	0.659	0.608
0.70	2	10	18	0.434	0.402	0.724	0.698	0.634	0.683	0.705	0.711	0.660
0.70	2	10	19	0.477	0.456	0.771	0.746	0.683	0.731	0.754	0.760	0.709
0.70	2	10	20	0.519	0.509	0.813	0.790	0.731	0.775	0.799	0.803	0.754
0.70	2	10	21	0.561	0.564	0.849	0.829	0.775	0.815	0.837	0.841	0.795
0.70	2	20	1	0.050	0.008	0.056	0.050	0.049	0.050	0.050	0.050	0.050
0.70	2	20	2	0.053	0.009	0.061	0.057	0.056	0.055	0.055	0.055	0.055
0.70	2	20	3	0.061	0.013	0.080	0.074	0.071	0.072	0.072	0.072	0.071
0.70	2	20	4	0.073	0.021	0.107	0.100	0.096	0.097	0.101	0.099	0.097
0.70	2	20	5	0.094	0.035	0.149	0.141	0.134	0.135	0.140	0.139	0.134
0.70	2	20	6	0.118	0.054	0.203	0.191	0.182	0.185	0.193	0.190	0.183
0.70	2	20	7	0.151	0.079	0.268	0.255	0.243	0.247	0.257	0.254	0.243
0.70	2	20	8	0.190	0.115	0.341	0.328	0.311	0.320	0.331	0.327	0.313
0.70	2	20	9	0.233	0.160	0.422	0.407	0.386	0.400	0.409	0.408	0.391
0.70	2	20	10	0.283	0.216	0.508	0.490	0.467	0.482	0.494	0.493	0.473
0.70	2	20	11	0.339	0.284	0.591	0.575	0.550	0.568	0.580	0.578	0.556
0.70	2	20	12	0.396	0.357	0.670	0.658	0.629	0.650	0.663	0.660	0.637
0.70	2	20	13	0.457	0.435	0.746	0.731	0.707	0.724	0.740	0.735	0.713
0.70	2	20	14	0.518	0.518	0.809	0.796	0.775	0.791	0.803	0.801	0.779
0.70	2	20	15	0.577	0.601	0.863	0.853	0.833	0.845	0.858	0.856	0.836
0.70	2	20	16	0.634	0.679	0.904	0.897	0.880	0.890	0.900	0.899	0.883
0.70	2	20	17	0.690	0.750	0.936	0.930	0.917	0.926	0.934	0.932	0.919
0.70	2	20	18	0.742	0.811	0.960	0.956	0.944	0.951	0.958	0.956	0.946
0.70	2	20	19	0.791	0.863	0.975	0.972	0.965	0.969	0.975	0.973	0.965
0.70	2	20	20	0.834	0.904	0.985	0.983	0.978	0.981	0.985	0.984	0.979
0.70	2	20	21	0.870	0.935	0.991	0.991	0.987	0.989	0.991	0.991	0.987

Table 1: Power simulation (see Section 2 for details), $\rho = 0.7$, $J = 2$, $I = 10, 20$

					ps alloc						theoretical	
				no ps alloc	1-way analysis			ps alloc	ancovar		power	
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.70	2	40	1	0.049	0.006	0.053	0.049	0.051	0.049	0.050	0.050	0.050
0.70	2	40	2	0.055	0.009	0.064	0.060	0.060	0.062	0.060	0.061	0.061
0.70	2	40	3	0.072	0.018	0.097	0.094	0.092	0.096	0.094	0.095	0.094
0.70	2	40	4	0.102	0.035	0.155	0.150	0.146	0.150	0.151	0.153	0.150
0.70	2	40	5	0.143	0.067	0.239	0.232	0.226	0.232	0.233	0.236	0.231
0.70	2	40	6	0.198	0.116	0.343	0.336	0.329	0.337	0.337	0.340	0.333
0.70	2	40	7	0.265	0.186	0.461	0.455	0.443	0.455	0.453	0.458	0.449
0.70	2	40	8	0.339	0.279	0.584	0.575	0.561	0.575	0.580	0.581	0.571
0.70	2	40	9	0.423	0.389	0.698	0.689	0.680	0.688	0.695	0.696	0.685
0.70	2	40	10	0.510	0.509	0.796	0.790	0.780	0.786	0.794	0.795	0.785
0.70	2	40	11	0.597	0.627	0.870	0.865	0.857	0.866	0.868	0.871	0.863
0.70	2	40	12	0.677	0.735	0.924	0.921	0.914	0.921	0.924	0.925	0.919
0.70	2	40	13	0.751	0.824	0.961	0.959	0.954	0.958	0.960	0.960	0.956
0.70	2	40	14	0.817	0.890	0.982	0.980	0.978	0.978	0.981	0.980	0.978
0.70	2	40	15	0.869	0.938	0.992	0.991	0.990	0.990	0.991	0.991	0.990
0.70	2	40	16	0.910	0.968	0.996	0.996	0.995	0.996	0.996	0.996	0.996
0.70	2	40	17	0.941	0.984	0.999	0.999	0.998	0.999	0.998	0.999	0.998
0.70	2	40	18	0.963	0.993	1.000	1.000	0.999	1.000	1.000	1.000	0.999
0.70	2	40	19	0.977	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.70	2	40	20	0.987	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.70	2	40	21	0.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 1: Power simulation (see Section 2 for details), $\rho = 0.7$, $J = 2$, $I = 40$

					ps alloc						theoretical	
				no ps alloc	1-way analysis			ps alloc	ancovar		power	
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.70	5	3	1	0.048	0.006	0.139	0.051	0.050	0.051	0.049	0.050	0.050
0.70	5	3	2	0.048	0.006	0.139	0.052	0.050	0.052	0.049	0.050	0.050
0.70	5	3	3	0.049	0.006	0.139	0.052	0.051	0.052	0.050	0.051	0.051
0.70	5	3	4	0.050	0.006	0.141	0.054	0.052	0.053	0.051	0.052	0.052
0.70	5	3	5	0.050	0.006	0.144	0.055	0.053	0.055	0.053	0.054	0.054
0.70	5	3	6	0.051	0.007	0.146	0.057	0.054	0.057	0.055	0.056	0.056
0.70	5	3	7	0.053	0.007	0.149	0.060	0.056	0.059	0.058	0.059	0.058
0.70	5	3	8	0.055	0.007	0.154	0.063	0.059	0.062	0.061	0.062	0.061
0.70	5	3	9	0.057	0.008	0.158	0.067	0.062	0.066	0.065	0.066	0.065
0.70	5	3	10	0.058	0.009	0.164	0.070	0.065	0.070	0.069	0.070	0.069
0.70	5	3	11	0.061	0.010	0.170	0.075	0.069	0.074	0.074	0.075	0.073
0.70	5	3	12	0.063	0.011	0.177	0.081	0.073	0.079	0.080	0.081	0.078
0.70	5	3	13	0.066	0.011	0.184	0.086	0.078	0.085	0.085	0.087	0.084
0.70	5	3	14	0.069	0.013	0.193	0.092	0.083	0.091	0.092	0.094	0.090
0.70	5	3	15	0.073	0.014	0.202	0.099	0.089	0.097	0.098	0.102	0.097
0.70	5	3	16	0.077	0.015	0.211	0.107	0.095	0.105	0.107	0.111	0.105
0.70	5	3	17	0.081	0.017	0.221	0.116	0.101	0.114	0.116	0.120	0.113
0.70	5	3	18	0.086	0.019	0.233	0.124	0.109	0.122	0.127	0.130	0.122
0.70	5	3	19	0.091	0.021	0.244	0.134	0.116	0.131	0.137	0.141	0.132
0.70	5	3	20	0.097	0.024	0.256	0.144	0.124	0.141	0.147	0.153	0.143
0.70	5	3	21	0.103	0.026	0.269	0.155	0.133	0.151	0.159	0.166	0.154
0.70	5	5	1	0.047	0.003	0.085	0.050	0.049	0.051	0.049	0.050	0.050
0.70	5	5	2	0.047	0.003	0.086	0.051	0.050	0.051	0.050	0.050	0.050
0.70	5	5	3	0.048	0.003	0.087	0.052	0.051	0.053	0.052	0.052	0.052
0.70	5	5	4	0.050	0.004	0.091	0.054	0.053	0.056	0.054	0.054	0.054
0.70	5	5	5	0.052	0.004	0.095	0.057	0.056	0.059	0.057	0.058	0.057
0.70	5	5	6	0.054	0.004	0.101	0.062	0.060	0.063	0.062	0.062	0.062
0.70	5	5	7	0.057	0.005	0.107	0.067	0.064	0.068	0.068	0.068	0.067
0.70	5	5	8	0.060	0.005	0.115	0.074	0.071	0.075	0.075	0.075	0.074
0.70	5	5	9	0.064	0.006	0.126	0.081	0.078	0.083	0.084	0.083	0.082
0.70	5	5	10	0.069	0.007	0.137	0.090	0.085	0.092	0.093	0.093	0.091
0.70	5	5	11	0.075	0.009	0.148	0.100	0.095	0.104	0.103	0.104	0.101
0.70	5	5	12	0.080	0.010	0.162	0.111	0.105	0.116	0.115	0.117	0.113
0.70	5	5	13	0.087	0.012	0.178	0.125	0.117	0.129	0.130	0.132	0.127
0.70	5	5	14	0.095	0.014	0.196	0.139	0.131	0.144	0.146	0.148	0.142
0.70	5	5	15	0.103	0.018	0.217	0.156	0.147	0.162	0.164	0.166	0.159
0.70	5	5	16	0.111	0.021	0.238	0.173	0.163	0.181	0.184	0.186	0.178
0.70	5	5	17	0.122	0.025	0.261	0.193	0.181	0.201	0.206	0.208	0.198
0.70	5	5	18	0.132	0.030	0.284	0.215	0.200	0.223	0.231	0.231	0.221
0.70	5	5	19	0.143	0.035	0.310	0.240	0.221	0.246	0.255	0.257	0.244
0.70	5	5	20	0.157	0.041	0.338	0.266	0.244	0.272	0.283	0.284	0.270
0.70	5	5	21	0.170	0.048	0.367	0.294	0.269	0.298	0.312	0.313	0.297

Table 1: Power simulation (see Section 2 for details), $\rho = 0.7$, $J = 5$, $I = 3, 5$

				no ps alloc	ps alloc 1-way analysis			ps alloc	ancovar		theoretical power	
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.70	5	10	1	0.050	0.002	0.066	0.051	0.051	0.049	0.052	0.050	0.050
0.70	5	10	2	0.050	0.002	0.067	0.052	0.052	0.049	0.053	0.051	0.051
0.70	5	10	3	0.052	0.002	0.070	0.054	0.055	0.053	0.056	0.054	0.054
0.70	5	10	4	0.056	0.002	0.076	0.060	0.060	0.059	0.061	0.060	0.060
0.70	5	10	5	0.059	0.003	0.086	0.068	0.067	0.067	0.069	0.068	0.068
0.70	5	10	6	0.065	0.004	0.098	0.078	0.078	0.078	0.079	0.079	0.078
0.70	5	10	7	0.072	0.005	0.114	0.092	0.091	0.092	0.094	0.093	0.092
0.70	5	10	8	0.082	0.007	0.131	0.108	0.107	0.110	0.111	0.111	0.109
0.70	5	10	9	0.091	0.009	0.152	0.128	0.127	0.131	0.131	0.132	0.130
0.70	5	10	10	0.102	0.012	0.179	0.152	0.148	0.156	0.156	0.157	0.154
0.70	5	10	11	0.116	0.016	0.210	0.181	0.176	0.186	0.185	0.186	0.183
0.70	5	10	12	0.132	0.021	0.244	0.214	0.207	0.218	0.218	0.220	0.215
0.70	5	10	13	0.148	0.028	0.281	0.251	0.241	0.256	0.257	0.258	0.252
0.70	5	10	14	0.168	0.037	0.324	0.290	0.281	0.296	0.297	0.300	0.293
0.70	5	10	15	0.189	0.048	0.370	0.335	0.323	0.342	0.344	0.345	0.337
0.70	5	10	16	0.212	0.062	0.419	0.381	0.367	0.389	0.393	0.394	0.385
0.70	5	10	17	0.238	0.080	0.469	0.430	0.417	0.441	0.445	0.445	0.435
0.70	5	10	18	0.266	0.101	0.519	0.479	0.467	0.493	0.496	0.498	0.486
0.70	5	10	19	0.295	0.126	0.571	0.534	0.517	0.544	0.549	0.551	0.538
0.70	5	10	20	0.327	0.154	0.620	0.586	0.570	0.595	0.601	0.603	0.590
0.70	5	10	21	0.359	0.186	0.671	0.636	0.618	0.646	0.654	0.654	0.641
0.70	5	20	1	0.049	0.001	0.057	0.049	0.050	0.049	0.050	0.050	0.050
0.70	5	20	2	0.051	0.001	0.059	0.052	0.052	0.052	0.052	0.052	0.052
0.70	5	20	3	0.055	0.002	0.066	0.059	0.059	0.058	0.061	0.059	0.059
0.70	5	20	4	0.062	0.003	0.079	0.072	0.071	0.070	0.072	0.072	0.072
0.70	5	20	5	0.069	0.004	0.099	0.088	0.089	0.087	0.090	0.090	0.090
0.70	5	20	6	0.080	0.006	0.124	0.112	0.113	0.112	0.115	0.116	0.115
0.70	5	20	7	0.095	0.009	0.158	0.146	0.145	0.145	0.147	0.149	0.148
0.70	5	20	8	0.113	0.015	0.202	0.187	0.185	0.188	0.190	0.192	0.190
0.70	5	20	9	0.137	0.023	0.256	0.237	0.236	0.239	0.242	0.243	0.240
0.70	5	20	10	0.165	0.035	0.314	0.296	0.293	0.298	0.301	0.303	0.300
0.70	5	20	11	0.197	0.052	0.383	0.364	0.360	0.365	0.369	0.371	0.367
0.70	5	20	12	0.233	0.074	0.457	0.435	0.431	0.438	0.442	0.444	0.439
0.70	5	20	13	0.277	0.105	0.534	0.510	0.508	0.514	0.520	0.521	0.515
0.70	5	20	14	0.323	0.146	0.608	0.588	0.583	0.591	0.597	0.597	0.591
0.70	5	20	15	0.371	0.194	0.681	0.664	0.658	0.663	0.671	0.671	0.665
0.70	5	20	16	0.423	0.254	0.750	0.731	0.727	0.733	0.742	0.740	0.734
0.70	5	20	17	0.479	0.320	0.809	0.793	0.788	0.792	0.801	0.800	0.795
0.70	5	20	18	0.533	0.393	0.858	0.846	0.842	0.846	0.852	0.852	0.847
0.70	5	20	19	0.588	0.471	0.898	0.891	0.884	0.891	0.895	0.894	0.889
0.70	5	20	20	0.640	0.551	0.928	0.923	0.918	0.924	0.926	0.926	0.923
0.70	5	20	21	0.692	0.628	0.952	0.947	0.944	0.949	0.951	0.951	0.948

Table 1: Power simulation (see Section 2 for details), $\rho = 0.7$, $J = 5$, $I = 10, 20$

					ps alloc						theoretical	
				no ps alloc	1-way analysis			ps alloc	ancovar		power	
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.70	5	40	1	0.048	0.001	0.053	0.049	0.050	0.049	0.050	0.050	0.050
0.70	5	40	2	0.051	0.001	0.057	0.054	0.054	0.053	0.054	0.055	0.055
0.70	5	40	3	0.058	0.002	0.073	0.069	0.070	0.069	0.069	0.070	0.070
0.70	5	40	4	0.072	0.004	0.101	0.095	0.097	0.097	0.097	0.097	0.097
0.70	5	40	5	0.091	0.008	0.142	0.136	0.137	0.139	0.138	0.139	0.139
0.70	5	40	6	0.117	0.015	0.203	0.195	0.197	0.199	0.197	0.199	0.198
0.70	5	40	7	0.153	0.029	0.284	0.274	0.274	0.277	0.278	0.278	0.276
0.70	5	40	8	0.198	0.050	0.378	0.369	0.368	0.371	0.372	0.373	0.371
0.70	5	40	9	0.251	0.085	0.489	0.477	0.475	0.477	0.481	0.479	0.476
0.70	5	40	10	0.316	0.137	0.597	0.588	0.585	0.588	0.593	0.589	0.586
0.70	5	40	11	0.385	0.209	0.701	0.691	0.690	0.691	0.695	0.694	0.691
0.70	5	40	12	0.462	0.302	0.788	0.782	0.779	0.781	0.786	0.786	0.783
0.70	5	40	13	0.541	0.411	0.864	0.858	0.857	0.854	0.861	0.860	0.858
0.70	5	40	14	0.618	0.526	0.917	0.913	0.912	0.910	0.916	0.915	0.913
0.70	5	40	15	0.691	0.638	0.954	0.951	0.950	0.948	0.953	0.952	0.951
0.70	5	40	16	0.760	0.739	0.976	0.974	0.974	0.973	0.975	0.975	0.974
0.70	5	40	17	0.819	0.826	0.988	0.987	0.987	0.986	0.988	0.988	0.987
0.70	5	40	18	0.870	0.892	0.995	0.995	0.994	0.994	0.995	0.995	0.994
0.70	5	40	19	0.907	0.938	0.998	0.998	0.998	0.997	0.998	0.998	0.998
0.70	5	40	20	0.937	0.966	0.999	0.999	0.999	0.999	0.999	0.999	0.999
0.70	5	40	21	0.959	0.983	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 1: Power simulation (see Section 2 for details), $\rho = 0.7$, $J = 5$, $I = 40$

					ps alloc						theoretical	
				no ps alloc	1-way analysis			ps alloc	ancovar		power	
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.90	2	3	1	0.050	0.003	0.255	0.086	0.046	0.050	0.050	0.050	0.050
0.90	2	3	2	0.050	0.003	0.257	0.087	0.046	0.051	0.052	0.051	0.051
0.90	2	3	3	0.051	0.004	0.261	0.092	0.048	0.054	0.055	0.056	0.054
0.90	2	3	4	0.052	0.004	0.269	0.099	0.050	0.058	0.061	0.063	0.058
0.90	2	3	5	0.054	0.005	0.280	0.109	0.053	0.064	0.068	0.073	0.065
0.90	2	3	6	0.056	0.006	0.291	0.122	0.059	0.073	0.079	0.085	0.073
0.90	2	3	7	0.060	0.007	0.305	0.137	0.064	0.084	0.092	0.101	0.082
0.90	2	3	8	0.063	0.009	0.323	0.156	0.070	0.097	0.108	0.120	0.094
0.90	2	3	9	0.066	0.011	0.343	0.176	0.077	0.111	0.126	0.141	0.107
0.90	2	3	10	0.070	0.013	0.367	0.199	0.085	0.128	0.146	0.166	0.121
0.90	2	3	11	0.074	0.016	0.391	0.222	0.094	0.147	0.167	0.193	0.137
0.90	2	3	12	0.080	0.019	0.413	0.247	0.103	0.168	0.191	0.223	0.154
0.90	2	3	13	0.086	0.023	0.438	0.275	0.112	0.190	0.216	0.256	0.173
0.90	2	3	14	0.092	0.027	0.463	0.303	0.123	0.214	0.244	0.290	0.193
0.90	2	3	15	0.100	0.031	0.491	0.334	0.134	0.239	0.274	0.327	0.213
0.90	2	3	16	0.108	0.036	0.518	0.365	0.147	0.266	0.305	0.365	0.235
0.90	2	3	17	0.117	0.042	0.545	0.396	0.159	0.294	0.336	0.405	0.257
0.90	2	3	18	0.126	0.049	0.573	0.428	0.171	0.322	0.369	0.445	0.281
0.90	2	3	19	0.135	0.057	0.600	0.458	0.184	0.351	0.404	0.486	0.304
0.90	2	3	20	0.146	0.065	0.627	0.490	0.199	0.381	0.439	0.526	0.329
0.90	2	3	21	0.156	0.072	0.653	0.522	0.213	0.410	0.472	0.566	0.354
0.90	2	5	1	0.051	0.001	0.140	0.083	0.046	0.049	0.048	0.050	0.050
0.90	2	5	2	0.052	0.001	0.144	0.086	0.048	0.053	0.050	0.053	0.052
0.90	2	5	3	0.053	0.001	0.155	0.095	0.053	0.060	0.059	0.062	0.059
0.90	2	5	4	0.056	0.002	0.170	0.110	0.060	0.072	0.075	0.077	0.071
0.90	2	5	5	0.060	0.003	0.194	0.131	0.070	0.091	0.096	0.098	0.088
0.90	2	5	6	0.065	0.004	0.223	0.158	0.085	0.116	0.122	0.126	0.109
0.90	2	5	7	0.071	0.006	0.257	0.190	0.103	0.146	0.157	0.161	0.136
0.90	2	5	8	0.078	0.008	0.297	0.228	0.124	0.181	0.196	0.202	0.167
0.90	2	5	9	0.087	0.011	0.339	0.269	0.148	0.221	0.240	0.249	0.203
0.90	2	5	10	0.097	0.015	0.385	0.314	0.174	0.267	0.291	0.302	0.243
0.90	2	5	11	0.109	0.020	0.436	0.360	0.204	0.318	0.346	0.359	0.287
0.90	2	5	12	0.121	0.026	0.487	0.411	0.237	0.371	0.404	0.419	0.335
0.90	2	5	13	0.134	0.034	0.538	0.464	0.274	0.428	0.464	0.482	0.385
0.90	2	5	14	0.149	0.044	0.588	0.515	0.311	0.483	0.526	0.545	0.436
0.90	2	5	15	0.165	0.054	0.639	0.568	0.349	0.542	0.588	0.606	0.489
0.90	2	5	16	0.183	0.069	0.685	0.620	0.390	0.599	0.645	0.665	0.541
0.90	2	5	17	0.203	0.085	0.728	0.669	0.432	0.653	0.699	0.720	0.592
0.90	2	5	18	0.222	0.103	0.768	0.714	0.474	0.704	0.750	0.770	0.642
0.90	2	5	19	0.242	0.125	0.806	0.755	0.515	0.749	0.795	0.815	0.689
0.90	2	5	20	0.263	0.147	0.838	0.793	0.557	0.790	0.835	0.854	0.732
0.90	2	5	21	0.285	0.173	0.866	0.826	0.596	0.827	0.870	0.887	0.773

Table 1: Power simulation (see Section 2 for details), $\rho = 0.9$, $J = 2$, $I = 3, 5$

				no ps alloc	ps alloc 1-way analysis			ps alloc	ancovar		theoretical power	
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.90	2	10	1	0.052	0.000	0.087	0.067	0.048	0.050	0.049	0.050	0.050
0.90	2	10	2	0.052	0.000	0.095	0.074	0.053	0.056	0.057	0.057	0.056
0.90	2	10	3	0.056	0.001	0.119	0.095	0.069	0.076	0.077	0.077	0.075
0.90	2	10	4	0.062	0.001	0.157	0.131	0.095	0.110	0.114	0.113	0.106
0.90	2	10	5	0.071	0.002	0.209	0.179	0.131	0.158	0.163	0.163	0.151
0.90	2	10	6	0.083	0.004	0.275	0.242	0.182	0.220	0.228	0.229	0.210
0.90	2	10	7	0.099	0.006	0.349	0.313	0.240	0.296	0.305	0.308	0.280
0.90	2	10	8	0.116	0.011	0.435	0.397	0.308	0.383	0.396	0.397	0.362
0.90	2	10	9	0.136	0.018	0.523	0.483	0.382	0.473	0.490	0.493	0.449
0.90	2	10	10	0.159	0.028	0.610	0.571	0.465	0.564	0.587	0.589	0.540
0.90	2	10	11	0.186	0.046	0.689	0.654	0.545	0.652	0.677	0.680	0.628
0.90	2	10	12	0.215	0.067	0.762	0.729	0.624	0.736	0.760	0.761	0.710
0.90	2	10	13	0.247	0.096	0.823	0.796	0.697	0.805	0.827	0.829	0.782
0.90	2	10	14	0.279	0.134	0.874	0.852	0.761	0.861	0.882	0.883	0.842
0.90	2	10	15	0.314	0.181	0.914	0.897	0.818	0.907	0.923	0.924	0.890
0.90	2	10	16	0.352	0.235	0.944	0.929	0.865	0.939	0.953	0.953	0.927
0.90	2	10	17	0.392	0.298	0.964	0.955	0.903	0.960	0.972	0.972	0.953
0.90	2	10	18	0.434	0.365	0.978	0.972	0.932	0.976	0.984	0.985	0.971
0.90	2	10	19	0.477	0.437	0.988	0.983	0.956	0.986	0.991	0.992	0.983
0.90	2	10	20	0.519	0.512	0.993	0.990	0.970	0.992	0.996	0.996	0.991
0.90	2	10	21	0.560	0.588	0.996	0.994	0.981	0.996	0.997	0.998	0.995
0.90	2	20	1	0.050	0.000	0.065	0.057	0.050	0.050	0.051	0.050	0.050
0.90	2	20	2	0.052	0.000	0.082	0.073	0.063	0.064	0.066	0.064	0.064
0.90	2	20	3	0.060	0.000	0.130	0.118	0.103	0.108	0.112	0.109	0.106
0.90	2	20	4	0.073	0.001	0.209	0.194	0.169	0.185	0.188	0.186	0.179
0.90	2	20	5	0.092	0.002	0.314	0.295	0.262	0.288	0.294	0.293	0.281
0.90	2	20	6	0.119	0.007	0.440	0.419	0.377	0.415	0.424	0.424	0.406
0.90	2	20	7	0.151	0.017	0.577	0.554	0.509	0.553	0.564	0.564	0.542
0.90	2	20	8	0.191	0.035	0.703	0.683	0.637	0.684	0.698	0.697	0.673
0.90	2	20	9	0.235	0.066	0.807	0.793	0.749	0.797	0.805	0.807	0.786
0.90	2	20	10	0.286	0.118	0.884	0.875	0.840	0.881	0.887	0.889	0.872
0.90	2	20	11	0.342	0.191	0.938	0.930	0.906	0.938	0.943	0.942	0.930
0.90	2	20	12	0.399	0.287	0.968	0.965	0.950	0.970	0.973	0.973	0.966
0.90	2	20	13	0.459	0.398	0.986	0.984	0.974	0.987	0.988	0.989	0.985
0.90	2	20	14	0.522	0.519	0.995	0.994	0.988	0.995	0.996	0.996	0.994
0.90	2	20	15	0.582	0.641	0.998	0.998	0.996	0.998	0.999	0.999	0.998
0.90	2	20	16	0.641	0.747	0.999	0.999	0.999	1.000	1.000	1.000	0.999
0.90	2	20	17	0.695	0.834	1.000	1.000	0.999	1.000	1.000	1.000	1.000
0.90	2	20	18	0.747	0.898	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	2	20	19	0.795	0.942	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	2	20	20	0.836	0.969	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	2	20	21	0.871	0.985	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 1: Power simulation (see Section 2 for details), $\rho = 0.9$, $J = 2$, $I = 10, 20$

ρ	J	I	m	no ps alloc	ps alloc			ps alloc	ancovar		theoretical power	
					1-way analysis							
				1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.90	2	40	1	0.049	0.000	0.058	0.054	0.051	0.050	0.052	0.050	0.050
0.90	2	40	2	0.054	0.000	0.087	0.084	0.078	0.078	0.080	0.080	0.079
0.90	2	40	3	0.071	0.000	0.183	0.176	0.165	0.171	0.172	0.173	0.170
0.90	2	40	4	0.100	0.002	0.341	0.332	0.314	0.329	0.331	0.330	0.323
0.90	2	40	5	0.141	0.010	0.532	0.522	0.501	0.522	0.527	0.527	0.517
0.90	2	40	6	0.195	0.031	0.716	0.706	0.688	0.712	0.714	0.717	0.706
0.90	2	40	7	0.261	0.084	0.858	0.851	0.834	0.855	0.857	0.860	0.851
0.90	2	40	8	0.337	0.183	0.940	0.937	0.928	0.943	0.943	0.944	0.938
0.90	2	40	9	0.419	0.333	0.980	0.979	0.974	0.980	0.981	0.982	0.979
0.90	2	40	10	0.510	0.516	0.994	0.994	0.992	0.995	0.995	0.995	0.994
0.90	2	40	11	0.597	0.692	0.999	0.999	0.998	0.999	0.999	0.999	0.999
0.90	2	40	12	0.680	0.835	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	2	40	13	0.756	0.926	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	2	40	14	0.822	0.972	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	2	40	15	0.873	0.991	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	2	40	16	0.911	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	2	40	17	0.941	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	2	40	18	0.963	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	2	40	19	0.977	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	2	40	20	0.987	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	2	40	21	0.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 1: Power simulation (see Section 2 for details), $\rho = 0.9$, $J = 2$, $I = 40$

ρ	J	I	m	no ps alloc 1-way anal	ps alloc 1-way analysis no ρ	$\hat{\rho}$	true ρ	ps alloc 2-way anal	ancovar no ps alloc	ps alloc	theoretical power 1-way	2-way
0.90	5	3	1	0.048	0.000	0.343	0.111	0.046	0.052	0.050	0.050	0.050
0.90	5	3	2	0.048	0.000	0.342	0.113	0.046	0.052	0.051	0.051	0.051
0.90	5	3	3	0.048	0.000	0.344	0.116	0.048	0.054	0.053	0.053	0.052
0.90	5	3	4	0.049	0.000	0.348	0.119	0.049	0.057	0.056	0.056	0.055
0.90	5	3	5	0.049	0.000	0.355	0.124	0.051	0.062	0.060	0.061	0.060
0.90	5	3	6	0.051	0.000	0.361	0.131	0.054	0.067	0.066	0.067	0.065
0.90	5	3	7	0.053	0.000	0.370	0.139	0.058	0.073	0.074	0.074	0.072
0.90	5	3	8	0.054	0.000	0.379	0.150	0.063	0.083	0.082	0.084	0.081
0.90	5	3	9	0.055	0.000	0.391	0.161	0.067	0.093	0.094	0.095	0.091
0.90	5	3	10	0.057	0.000	0.404	0.174	0.073	0.103	0.106	0.109	0.103
0.90	5	3	11	0.060	0.000	0.419	0.190	0.079	0.117	0.121	0.124	0.117
0.90	5	3	12	0.062	0.001	0.434	0.208	0.086	0.133	0.137	0.142	0.132
0.90	5	3	13	0.065	0.001	0.450	0.226	0.094	0.150	0.157	0.161	0.150
0.90	5	3	14	0.068	0.001	0.467	0.246	0.104	0.171	0.177	0.184	0.170
0.90	5	3	15	0.072	0.001	0.485	0.267	0.114	0.193	0.201	0.208	0.192
0.90	5	3	16	0.076	0.001	0.502	0.289	0.125	0.215	0.228	0.235	0.216
0.90	5	3	17	0.080	0.001	0.521	0.314	0.137	0.242	0.256	0.265	0.242
0.90	5	3	18	0.084	0.002	0.540	0.340	0.151	0.271	0.286	0.296	0.270
0.90	5	3	19	0.089	0.002	0.561	0.367	0.165	0.301	0.316	0.330	0.300
0.90	5	3	20	0.095	0.002	0.581	0.394	0.181	0.333	0.349	0.365	0.332
0.90	5	3	21	0.100	0.003	0.603	0.423	0.197	0.365	0.383	0.402	0.365
0.90	5	5	1	0.047	0.000	0.193	0.095	0.049	0.051	0.050	0.050	0.050
0.90	5	5	2	0.047	0.000	0.195	0.097	0.050	0.052	0.052	0.051	0.051
0.90	5	5	3	0.048	0.000	0.202	0.101	0.052	0.056	0.057	0.055	0.055
0.90	5	5	4	0.049	0.000	0.212	0.110	0.057	0.062	0.064	0.062	0.061
0.90	5	5	5	0.050	0.000	0.226	0.121	0.063	0.071	0.074	0.072	0.071
0.90	5	5	6	0.053	0.000	0.244	0.134	0.070	0.084	0.087	0.085	0.083
0.90	5	5	7	0.056	0.000	0.265	0.152	0.081	0.100	0.104	0.102	0.099
0.90	5	5	8	0.060	0.000	0.290	0.173	0.095	0.120	0.124	0.124	0.120
0.90	5	5	9	0.063	0.000	0.318	0.197	0.111	0.144	0.149	0.150	0.144
0.90	5	5	10	0.068	0.000	0.349	0.226	0.129	0.174	0.180	0.181	0.173
0.90	5	5	11	0.073	0.000	0.385	0.258	0.149	0.208	0.216	0.216	0.207
0.90	5	5	12	0.080	0.000	0.423	0.294	0.174	0.247	0.254	0.257	0.245
0.90	5	5	13	0.087	0.000	0.463	0.336	0.202	0.290	0.297	0.303	0.288
0.90	5	5	14	0.095	0.001	0.503	0.377	0.233	0.338	0.345	0.353	0.335
0.90	5	5	15	0.103	0.001	0.546	0.421	0.268	0.390	0.398	0.406	0.386
0.90	5	5	16	0.111	0.001	0.588	0.468	0.305	0.443	0.453	0.462	0.439
0.90	5	5	17	0.120	0.002	0.629	0.516	0.345	0.499	0.511	0.519	0.494
0.90	5	5	18	0.131	0.002	0.671	0.563	0.385	0.555	0.568	0.576	0.549
0.90	5	5	19	0.143	0.004	0.711	0.610	0.431	0.611	0.624	0.632	0.604
0.90	5	5	20	0.155	0.005	0.749	0.657	0.474	0.663	0.677	0.686	0.658
0.90	5	5	21	0.170	0.006	0.784	0.699	0.520	0.714	0.727	0.736	0.708

Table 1: Power simulation (see Section 2 for details), $\rho = 0.9$, $J = 5$, $I = 3, 5$

				no ps alloc	ps alloc 1-way analysis			ps alloc	ancovar		theoretical power	
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.90	5	10	1	0.049	0.000	0.104	0.070	0.048	0.047	0.050	0.050	0.050
0.90	5	10	2	0.049	0.000	0.109	0.074	0.051	0.051	0.052	0.053	0.053
0.90	5	10	3	0.052	0.000	0.121	0.084	0.058	0.059	0.061	0.062	0.062
0.90	5	10	4	0.054	0.000	0.141	0.101	0.071	0.076	0.076	0.078	0.077
0.90	5	10	5	0.059	0.000	0.171	0.126	0.090	0.100	0.101	0.102	0.101
0.90	5	10	6	0.064	0.000	0.210	0.161	0.116	0.132	0.133	0.136	0.134
0.90	5	10	7	0.071	0.000	0.261	0.203	0.150	0.175	0.176	0.181	0.177
0.90	5	10	8	0.078	0.000	0.320	0.256	0.195	0.231	0.234	0.237	0.232
0.90	5	10	9	0.088	0.000	0.388	0.318	0.251	0.299	0.302	0.305	0.297
0.90	5	10	10	0.101	0.000	0.459	0.391	0.316	0.375	0.379	0.381	0.372
0.90	5	10	11	0.114	0.000	0.536	0.466	0.388	0.457	0.463	0.465	0.454
0.90	5	10	12	0.129	0.001	0.612	0.544	0.462	0.543	0.551	0.552	0.540
0.90	5	10	13	0.148	0.001	0.682	0.621	0.539	0.627	0.636	0.637	0.624
0.90	5	10	14	0.168	0.002	0.750	0.696	0.613	0.705	0.715	0.717	0.704
0.90	5	10	15	0.190	0.004	0.808	0.761	0.685	0.778	0.785	0.787	0.775
0.90	5	10	16	0.212	0.006	0.859	0.820	0.752	0.837	0.844	0.847	0.836
0.90	5	10	17	0.237	0.009	0.900	0.869	0.813	0.886	0.892	0.895	0.886
0.90	5	10	18	0.265	0.015	0.931	0.910	0.862	0.924	0.929	0.931	0.924
0.90	5	10	19	0.295	0.024	0.955	0.939	0.900	0.952	0.955	0.956	0.951
0.90	5	10	20	0.327	0.037	0.971	0.960	0.932	0.970	0.973	0.974	0.970
0.90	5	10	21	0.363	0.055	0.982	0.974	0.954	0.983	0.985	0.985	0.983
0.90	5	20	1	0.051	0.000	0.074	0.057	0.049	0.048	0.049	0.050	0.050
0.90	5	20	2	0.052	0.000	0.080	0.063	0.054	0.055	0.055	0.056	0.056
0.90	5	20	3	0.055	0.000	0.104	0.084	0.072	0.077	0.074	0.076	0.076
0.90	5	20	4	0.061	0.000	0.146	0.122	0.106	0.116	0.112	0.113	0.113
0.90	5	20	5	0.070	0.000	0.209	0.178	0.160	0.175	0.169	0.172	0.170
0.90	5	20	6	0.082	0.000	0.292	0.257	0.233	0.256	0.252	0.254	0.251
0.90	5	20	7	0.097	0.000	0.394	0.357	0.327	0.362	0.356	0.359	0.355
0.90	5	20	8	0.116	0.000	0.511	0.473	0.436	0.481	0.475	0.480	0.474
0.90	5	20	9	0.140	0.000	0.631	0.594	0.558	0.604	0.601	0.606	0.599
0.90	5	20	10	0.168	0.001	0.741	0.708	0.677	0.721	0.721	0.723	0.717
0.90	5	20	11	0.200	0.002	0.831	0.806	0.780	0.818	0.819	0.821	0.816
0.90	5	20	12	0.238	0.005	0.899	0.882	0.861	0.890	0.894	0.894	0.890
0.90	5	20	13	0.279	0.011	0.947	0.934	0.920	0.941	0.943	0.943	0.940
0.90	5	20	14	0.325	0.023	0.974	0.967	0.959	0.971	0.972	0.973	0.971
0.90	5	20	15	0.374	0.047	0.988	0.985	0.980	0.988	0.989	0.988	0.987
0.90	5	20	16	0.427	0.084	0.995	0.994	0.992	0.995	0.996	0.995	0.995
0.90	5	20	17	0.480	0.140	0.999	0.998	0.997	0.998	0.999	0.998	0.998
0.90	5	20	18	0.535	0.218	1.000	0.999	0.999	1.000	1.000	0.999	0.999
0.90	5	20	19	0.590	0.318	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	5	20	20	0.642	0.435	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	5	20	21	0.695	0.555	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 1: Power simulation (see Section 2 for details), $\rho = 0.9$, $J = 5$, $I = 10, 20$

					ps alloc						theoretical	
				no ps alloc	1-way analysis			ps alloc	ancovar		power	
ρ	J	I	m	1-way anal	no ρ	$\hat{\rho}$	true ρ	2-way anal	no ps alloc	ps alloc	1-way	2-way
0.90	5	40	1	0.049	0.000	0.060	0.053	0.050	0.050	0.050	0.050	0.050
0.90	5	40	2	0.052	0.000	0.073	0.066	0.062	0.063	0.063	0.063	0.063
0.90	5	40	3	0.059	0.000	0.121	0.110	0.104	0.107	0.105	0.107	0.107
0.90	5	40	4	0.071	0.000	0.210	0.193	0.185	0.191	0.191	0.194	0.193
0.90	5	40	5	0.089	0.000	0.345	0.326	0.314	0.325	0.327	0.328	0.327
0.90	5	40	6	0.116	0.000	0.514	0.492	0.480	0.495	0.498	0.500	0.497
0.90	5	40	7	0.150	0.000	0.689	0.669	0.657	0.671	0.677	0.677	0.674
0.90	5	40	8	0.194	0.001	0.829	0.817	0.806	0.818	0.823	0.823	0.821
0.90	5	40	9	0.249	0.005	0.922	0.914	0.908	0.917	0.919	0.920	0.918
0.90	5	40	10	0.313	0.016	0.970	0.967	0.964	0.969	0.971	0.970	0.969
0.90	5	40	11	0.385	0.047	0.991	0.990	0.988	0.991	0.991	0.991	0.991
0.90	5	40	12	0.459	0.112	0.998	0.998	0.997	0.998	0.998	0.998	0.998
0.90	5	40	13	0.538	0.230	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	5	40	14	0.617	0.390	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	5	40	15	0.693	0.573	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	5	40	16	0.760	0.749	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	5	40	17	0.817	0.873	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	5	40	18	0.868	0.945	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	5	40	19	0.907	0.981	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	5	40	20	0.937	0.995	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.90	5	40	21	0.960	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 1: Power simulation (see Section 2 for details), $\rho = 0.9$, $J = 5$, $I = 40$

			Incorrect Anova		Corrected Anova			Corrected Anocovar		
ρ	J	I	1-way	2-way	(uses z)	(uses t)	Incorrect Anocovar	(uses z)	(uses t)	MLE
0	2	3	0.949	0.949	0.866	0.945	0.952	0.859	0.957	0.767
		5	0.949	0.950	0.906	0.942	0.950	0.907	0.950	0.865
		10	0.949	0.953	0.930	0.948	0.951	0.930	0.947	0.914
		20	0.951	0.950	0.942	0.948	0.950	0.940	0.948	0.933
		40	0.947	0.944	0.940	0.944	0.947	0.943	0.947	0.938
		80	0.953	0.954	0.951	0.953	0.953	0.951	0.952	0.950
	3	3	0.952	0.950	0.896	0.944	0.953	0.895	0.956	0.808
		5	0.951	0.951	0.922	0.948	0.953	0.924	0.952	0.881
		10	0.953	0.952	0.942	0.951	0.954	0.943	0.953	0.925
		20	0.954	0.954	0.947	0.953	0.953	0.948	0.953	0.939
		40	0.945	0.944	0.941	0.944	0.945	0.942	0.944	0.938
		80	0.950	0.951	0.949	0.950	0.950	0.948	0.949	0.947
	5	3	0.949	0.950	0.918	0.948	0.952	0.919	0.951	0.842
		5	0.952	0.951	0.935	0.950	0.950	0.935	0.950	0.897
		10	0.952	0.952	0.945	0.951	0.953	0.946	0.952	0.930
		20	0.949	0.948	0.945	0.947	0.949	0.945	0.948	0.937
		40	0.949	0.948	0.946	0.947	0.949	0.947	0.949	0.943
		80	0.954	0.953	0.953	0.953	0.954	0.954	0.955	0.952
	7	3	0.947	0.948	0.924	0.945	0.947	0.924	0.946	0.852
		5	0.949	0.950	0.940	0.948	0.949	0.939	0.949	0.907
		10	0.955	0.954	0.948	0.953	0.955	0.950	0.955	0.934
		20	0.951	0.950	0.948	0.950	0.951	0.949	0.951	0.943
		40	0.953	0.954	0.952	0.953	0.953	0.952	0.953	0.949
		80	0.949	0.949	0.948	0.948	0.948	0.948	0.948	0.946
	9	3	0.956	0.956	0.940	0.955	0.956	0.940	0.955	0.870
		5	0.948	0.948	0.938	0.947	0.947	0.938	0.947	0.903
		10	0.952	0.951	0.948	0.951	0.952	0.949	0.952	0.932
		20	0.955	0.955	0.953	0.955	0.955	0.953	0.955	0.946
		40	0.950	0.950	0.949	0.950	0.949	0.949	0.949	0.945
		80	0.951	0.951	0.950	0.951	0.951	0.950	0.951	0.948
	11	3	0.950	0.950	0.936	0.949	0.949	0.937	0.948	0.868
		5	0.951	0.952	0.945	0.951	0.951	0.946	0.951	0.911
		10	0.952	0.952	0.950	0.952	0.952	0.949	0.952	0.937
		20	0.951	0.951	0.950	0.951	0.951	0.950	0.951	0.945
		40	0.949	0.948	0.947	0.948	0.949	0.948	0.949	0.946
		80	0.946	0.946	0.946	0.946	0.946	0.946	0.946	0.945
	20	3	0.949	0.948	0.941	0.948	0.947	0.940	0.947	0.878
		5	0.947	0.948	0.944	0.948	0.948	0.944	0.948	0.911
		10	0.953	0.952	0.951	0.952	0.953	0.951	0.953	0.935
		20	0.948	0.948	0.947	0.948	0.948	0.947	0.948	0.942
		40	0.949	0.949	0.949	0.949	0.949	0.949	0.949	0.946
		80	0.951	0.951	0.951	0.951	0.951	0.951	0.951	0.950

Table 2: Actual coverage of nominal 95% confidence intervals

			Incorrect Anova		Corrected Anova			Corrected Anocovar		
ρ	J	I	1-way	2-way	(uses z)	(uses t)	Incorrect Anocovar	(uses z)	(uses t)	MLE
.50	2	3	0.959	0.945	0.875	0.947	0.939	0.859	0.959	0.779
		5	0.960	0.937	0.911	0.947	0.931	0.911	0.949	0.864
		10	0.964	0.937	0.935	0.950	0.934	0.934	0.951	0.911
		20	0.960	0.933	0.941	0.948	0.928	0.939	0.948	0.927
		40	0.962	0.930	0.944	0.947	0.928	0.945	0.949	0.934
		80	0.963	0.929	0.946	0.948	0.927	0.945	0.947	0.937
	3	3	0.965	0.943	0.901	0.949	0.939	0.896	0.958	0.820
		5	0.969	0.944	0.930	0.952	0.940	0.928	0.956	0.888
		10	0.965	0.938	0.939	0.949	0.937	0.940	0.949	0.922
		20	0.967	0.937	0.943	0.948	0.938	0.946	0.950	0.935
		40	0.969	0.937	0.948	0.950	0.938	0.948	0.951	0.941
		80	0.968	0.937	0.948	0.949	0.936	0.948	0.950	0.943
	5	3	0.971	0.944	0.921	0.950	0.942	0.919	0.952	0.844
		5	0.971	0.948	0.942	0.955	0.944	0.940	0.953	0.904
		10	0.969	0.941	0.942	0.949	0.940	0.941	0.949	0.925
		20	0.971	0.941	0.946	0.948	0.940	0.945	0.949	0.937
		40	0.971	0.941	0.947	0.948	0.940	0.947	0.948	0.941
		80	0.973	0.940	0.948	0.949	0.940	0.949	0.949	0.945
	7	3	0.971	0.946	0.930	0.950	0.943	0.928	0.951	0.854
		5	0.974	0.947	0.942	0.952	0.948	0.942	0.952	0.906
		10	0.972	0.945	0.945	0.950	0.944	0.946	0.949	0.930
		20	0.974	0.944	0.948	0.950	0.945	0.948	0.950	0.940
		40	0.973	0.945	0.949	0.950	0.945	0.949	0.950	0.946
		80	0.975	0.944	0.951	0.951	0.945	0.950	0.951	0.948
	9	3	0.974	0.950	0.939	0.953	0.949	0.937	0.955	0.866
		5	0.975	0.948	0.946	0.952	0.947	0.944	0.951	0.913
		10	0.974	0.945	0.946	0.949	0.945	0.946	0.950	0.932
		20	0.971	0.944	0.946	0.948	0.943	0.946	0.948	0.939
		40	0.973	0.946	0.948	0.949	0.945	0.948	0.949	0.945
		80	0.975	0.947	0.951	0.951	0.947	0.950	0.951	0.948
	11	3	0.976	0.949	0.941	0.952	0.948	0.939	0.952	0.872
		5	0.976	0.948	0.944	0.951	0.948	0.946	0.951	0.911
		10	0.975	0.948	0.949	0.952	0.947	0.948	0.951	0.933
		20	0.972	0.944	0.946	0.947	0.943	0.946	0.947	0.939
		40	0.976	0.949	0.951	0.952	0.948	0.952	0.952	0.948
		80	0.976	0.950	0.953	0.954	0.950	0.953	0.954	0.951
	20	3	0.977	0.947	0.942	0.949	0.946	0.941	0.948	0.879
		5	0.979	0.955	0.952	0.957	0.954	0.953	0.955	0.921
		10	0.977	0.950	0.951	0.951	0.950	0.951	0.952	0.936
		20	0.976	0.947	0.948	0.949	0.947	0.947	0.948	0.943
		40	0.976	0.948	0.950	0.950	0.949	0.951	0.951	0.947
		80	0.975	0.951	0.952	0.952	0.950	0.953	0.953	0.950

Table 2 continued: Actual coverage of nominal 95% confidence intervals

ρ	J	I	Incorrect Anova 1-way	Incorrect Anova 2-way	Corrected Anova (uses z)	Corrected Anova (uses t)	Incorrect Anocovar	Corrected Anocovar (uses z)	Corrected Anocovar (uses t)	MLE
.60	2	3	0.965	0.938	0.885	0.953	0.928	0.864	0.962	0.785
		5	0.967	0.932	0.916	0.950	0.924	0.911	0.954	0.863
		10	0.971	0.928	0.939	0.954	0.923	0.936	0.951	0.911
		20	0.969	0.920	0.943	0.950	0.918	0.941	0.948	0.924
		40	0.968	0.915	0.945	0.948	0.914	0.944	0.948	0.931
		80	0.969	0.916	0.947	0.949	0.916	0.947	0.949	0.936
	3	3	0.971	0.944	0.910	0.954	0.941	0.908	0.960	0.829
		5	0.975	0.941	0.935	0.956	0.935	0.932	0.956	0.890
		10	0.975	0.933	0.941	0.951	0.925	0.938	0.951	0.918
		20	0.978	0.931	0.948	0.953	0.931	0.949	0.954	0.936
		40	0.974	0.923	0.945	0.948	0.925	0.946	0.950	0.937
		80	0.973	0.931	0.950	0.951	0.931	0.950	0.951	0.944
	5	3	0.977	0.942	0.925	0.953	0.939	0.923	0.955	0.848
		5	0.979	0.941	0.940	0.953	0.941	0.941	0.955	0.904
		10	0.979	0.937	0.943	0.950	0.936	0.944	0.950	0.925
		20	0.979	0.937	0.949	0.952	0.938	0.948	0.951	0.939
		40	0.979	0.937	0.949	0.950	0.937	0.949	0.950	0.944
		80	0.980	0.937	0.948	0.949	0.936	0.948	0.949	0.945
	7	3	0.978	0.944	0.932	0.952	0.942	0.931	0.952	0.861
		5	0.983	0.942	0.941	0.953	0.943	0.943	0.953	0.910
		10	0.981	0.941	0.946	0.950	0.942	0.947	0.951	0.931
		20	0.981	0.942	0.949	0.950	0.940	0.948	0.949	0.940
		40	0.981	0.941	0.950	0.951	0.942	0.950	0.951	0.946
		80	0.980	0.940	0.948	0.949	0.939	0.948	0.949	0.946
	9	3	0.982	0.946	0.936	0.951	0.944	0.935	0.952	0.870
		5	0.983	0.942	0.942	0.949	0.943	0.943	0.950	0.912
		10	0.984	0.944	0.948	0.951	0.943	0.948	0.951	0.934
		20	0.982	0.943	0.949	0.951	0.943	0.949	0.950	0.942
		40	0.982	0.947	0.953	0.954	0.947	0.953	0.954	0.949
		80	0.983	0.945	0.952	0.952	0.945	0.951	0.952	0.949
	11	3	0.980	0.943	0.936	0.948	0.939	0.934	0.946	0.869
		5	0.983	0.943	0.942	0.950	0.942	0.942	0.951	0.908
		10	0.984	0.946	0.950	0.953	0.946	0.950	0.953	0.933
		20	0.982	0.943	0.948	0.949	0.944	0.948	0.949	0.942
		40	0.986	0.942	0.947	0.948	0.944	0.949	0.949	0.945
		80	0.983	0.946	0.952	0.952	0.946	0.951	0.951	0.949
	20	3	0.986	0.951	0.948	0.955	0.951	0.948	0.954	0.888
		5	0.986	0.950	0.949	0.953	0.949	0.949	0.952	0.918
		10	0.986	0.950	0.952	0.954	0.951	0.953	0.955	0.940
		20	0.987	0.950	0.952	0.953	0.950	0.952	0.953	0.946
		40	0.986	0.948	0.950	0.951	0.949	0.952	0.952	0.948
		80	0.985	0.945	0.949	0.949	0.946	0.949	0.950	0.948

Table 2 continued: Actual coverage of nominal 95% confidence intervals

			Incorrect Anova		Corrected Anova			Corrected Anocovar		
ρ	J	I	1-way	2-way	(uses z)	(uses t)	Incorrect Anocovar	(uses z)	(uses t)	MLE
.70	2	3	0.970	0.940	0.890	0.956	0.921	0.872	0.965	0.795
		5	0.974	0.922	0.923	0.959	0.909	0.918	0.958	0.867
		10	0.975	0.911	0.939	0.954	0.900	0.935	0.954	0.906
		20	0.978	0.904	0.948	0.956	0.898	0.947	0.954	0.924
		40	0.977	0.892	0.949	0.952	0.891	0.949	0.953	0.929
		80	0.974	0.893	0.947	0.949	0.891	0.948	0.950	0.931
	3	3	0.978	0.935	0.914	0.958	0.922	0.906	0.963	0.831
		5	0.983	0.931	0.936	0.958	0.922	0.934	0.957	0.887
		10	0.980	0.920	0.943	0.951	0.915	0.941	0.950	0.917
		20	0.982	0.913	0.945	0.949	0.910	0.944	0.949	0.928
		40	0.985	0.914	0.951	0.953	0.915	0.950	0.952	0.938
		80	0.983	0.914	0.950	0.951	0.916	0.950	0.951	0.941
	5	3	0.984	0.941	0.931	0.957	0.933	0.927	0.958	0.855
		5	0.987	0.933	0.940	0.953	0.928	0.938	0.952	0.900
		10	0.987	0.928	0.945	0.951	0.926	0.943	0.951	0.924
		20	0.988	0.925	0.948	0.952	0.925	0.946	0.949	0.935
		40	0.988	0.929	0.952	0.953	0.930	0.951	0.953	0.945
		80	0.988	0.926	0.949	0.950	0.927	0.948	0.949	0.944
	7	3	0.989	0.938	0.932	0.956	0.936	0.934	0.956	0.865
		5	0.990	0.942	0.946	0.954	0.939	0.945	0.956	0.913
		10	0.990	0.936	0.946	0.951	0.933	0.946	0.949	0.929
		20	0.990	0.932	0.949	0.951	0.930	0.948	0.950	0.938
		40	0.991	0.933	0.949	0.950	0.934	0.948	0.949	0.942
		80	0.990	0.935	0.952	0.952	0.936	0.951	0.952	0.948
	9	3	0.989	0.946	0.943	0.957	0.940	0.937	0.954	0.873
		5	0.993	0.945	0.949	0.957	0.943	0.948	0.957	0.916
		10	0.989	0.936	0.944	0.947	0.934	0.944	0.947	0.929
		20	0.991	0.937	0.949	0.950	0.937	0.950	0.952	0.941
		40	0.989	0.938	0.950	0.951	0.936	0.950	0.950	0.945
		80	0.992	0.941	0.952	0.952	0.941	0.952	0.952	0.949
	11	3	0.989	0.942	0.938	0.953	0.942	0.940	0.953	0.874
		5	0.992	0.942	0.945	0.953	0.941	0.946	0.953	0.913
		10	0.992	0.944	0.950	0.952	0.942	0.950	0.953	0.935
		20	0.991	0.940	0.950	0.951	0.941	0.950	0.952	0.942
		40	0.992	0.941	0.950	0.950	0.940	0.951	0.952	0.946
		80	0.991	0.939	0.949	0.949	0.938	0.948	0.948	0.946
	20	3	0.992	0.947	0.945	0.953	0.945	0.945	0.951	0.881
		5	0.991	0.943	0.946	0.949	0.943	0.945	0.949	0.912
		10	0.991	0.943	0.946	0.949	0.941	0.946	0.948	0.933
		20	0.991	0.940	0.945	0.946	0.940	0.945	0.945	0.938
		40	0.993	0.944	0.950	0.951	0.944	0.949	0.949	0.946
		80	0.993	0.943	0.949	0.949	0.944	0.949	0.949	0.947

Table 2 continued: Actual coverage of nominal 95% confidence intervals

			Incorrect Anova		Corrected Anova		Incorrect	Corrected Anocovar		
ρ	J	I	1-way	2-way	(uses z)	(uses t)	Anocovar	(uses z)	(uses t)	MLE
.80	2	3	0.976	0.931	0.898	0.962	0.901	0.873	0.963	0.802
		5	0.978	0.908	0.929	0.961	0.871	0.918	0.959	0.866
		10	0.983	0.881	0.943	0.959	0.864	0.941	0.956	0.906
		20	0.983	0.862	0.947	0.956	0.853	0.947	0.954	0.917
		40	0.981	0.848	0.947	0.951	0.846	0.946	0.950	0.920
		80	0.982	0.850	0.949	0.951	0.847	0.950	0.952	0.929
	3	3	0.986	0.929	0.924	0.966	0.904	0.915	0.969	0.838
		5	0.987	0.913	0.937	0.959	0.894	0.934	0.958	0.888
		10	0.989	0.894	0.945	0.954	0.882	0.943	0.953	0.916
		20	0.989	0.884	0.946	0.952	0.877	0.947	0.952	0.928
		40	0.991	0.878	0.946	0.949	0.876	0.947	0.950	0.929
		80	0.989	0.875	0.946	0.947	0.875	0.946	0.947	0.931
	5	3	0.990	0.933	0.933	0.959	0.914	0.927	0.959	0.861
		5	0.994	0.924	0.946	0.960	0.914	0.944	0.958	0.909
		10	0.993	0.914	0.946	0.953	0.907	0.947	0.953	0.926
		20	0.994	0.911	0.949	0.953	0.910	0.950	0.953	0.938
		40	0.994	0.903	0.946	0.947	0.904	0.945	0.947	0.937
		80	0.994	0.911	0.949	0.949	0.911	0.950	0.950	0.943
	7	3	0.994	0.937	0.940	0.958	0.923	0.938	0.957	0.873
		5	0.995	0.928	0.945	0.953	0.923	0.945	0.955	0.910
		10	0.997	0.929	0.950	0.953	0.923	0.950	0.953	0.932
		20	0.997	0.921	0.950	0.952	0.921	0.951	0.953	0.942
		40	0.996	0.925	0.956	0.957	0.925	0.955	0.956	0.949
		80	0.996	0.918	0.950	0.951	0.919	0.950	0.951	0.945
	9	3	0.996	0.938	0.942	0.957	0.930	0.941	0.956	0.876
		5	0.997	0.932	0.946	0.954	0.928	0.944	0.951	0.914
		10	0.998	0.931	0.951	0.953	0.926	0.949	0.953	0.931
		20	0.997	0.930	0.950	0.952	0.929	0.951	0.953	0.943
		40	0.997	0.930	0.951	0.952	0.931	0.951	0.952	0.947
		80	0.997	0.924	0.949	0.949	0.924	0.949	0.950	0.945
	11	3	0.996	0.940	0.943	0.954	0.932	0.941	0.954	0.882
		5	0.998	0.935	0.948	0.954	0.930	0.944	0.951	0.913
		10	0.998	0.933	0.949	0.952	0.933	0.947	0.950	0.935
		20	0.998	0.931	0.948	0.949	0.930	0.948	0.949	0.940
		40	0.997	0.931	0.950	0.950	0.931	0.949	0.950	0.946
		80	0.998	0.930	0.951	0.951	0.930	0.950	0.950	0.947
	20	3	0.996	0.942	0.944	0.951	0.941	0.947	0.954	0.888
		5	0.997	0.944	0.949	0.951	0.937	0.945	0.949	0.918
		10	0.998	0.942	0.950	0.951	0.941	0.949	0.951	0.937
		20	0.998	0.939	0.949	0.949	0.937	0.949	0.950	0.940
		40	0.999	0.942	0.952	0.952	0.941	0.950	0.951	0.947
		80	0.998	0.941	0.952	0.952	0.942	0.952	0.952	0.950

Table 2 continued: Actual coverage of nominal 95% confidence intervals

			Incorrect Anova		Corrected Anova			Corrected Anocovar		
ρ	J	I	1-way	2-way	(uses z)	(uses t)	Incorrect Anocovar	(uses z)	(uses t)	MLE
.90	2	3	0.982	0.920	0.904	0.968	0.828	0.875	0.966	0.797
		5	0.984	0.867	0.933	0.961	0.782	0.918	0.958	0.860
		10	0.986	0.799	0.940	0.954	0.747	0.933	0.950	0.885
		20	0.987	0.754	0.946	0.952	0.729	0.944	0.951	0.898
		40	0.989	0.749	0.950	0.953	0.737	0.948	0.951	0.911
		80	0.990	0.743	0.954	0.956	0.736	0.954	0.955	0.918
	3	3	0.991	0.908	0.926	0.967	0.839	0.914	0.967	0.843
		5	0.993	0.872	0.939	0.959	0.814	0.933	0.958	0.884
		10	0.995	0.835	0.948	0.958	0.802	0.948	0.958	0.914
		20	0.997	0.812	0.950	0.955	0.798	0.948	0.954	0.921
		40	0.995	0.800	0.949	0.951	0.792	0.950	0.952	0.925
		80	0.997	0.804	0.952	0.952	0.800	0.953	0.954	0.931
	5	3	0.998	0.921	0.944	0.967	0.870	0.939	0.966	0.880
		5	0.998	0.898	0.948	0.960	0.859	0.945	0.957	0.906
		10	0.998	0.871	0.950	0.955	0.850	0.948	0.954	0.924
		20	0.999	0.854	0.947	0.951	0.846	0.947	0.950	0.929
		40	0.999	0.853	0.951	0.952	0.846	0.950	0.952	0.939
		80	0.999	0.857	0.951	0.951	0.855	0.951	0.951	0.940
	7	3	0.998	0.924	0.943	0.960	0.883	0.939	0.957	0.881
		5	0.999	0.907	0.945	0.956	0.882	0.945	0.954	0.911
		10	0.999	0.888	0.947	0.951	0.878	0.947	0.952	0.928
		20	1.000	0.888	0.954	0.956	0.883	0.952	0.954	0.942
		40	1.000	0.884	0.949	0.950	0.882	0.949	0.950	0.941
		80	1.000	0.879	0.951	0.951	0.877	0.951	0.951	0.944
	9	3	0.999	0.934	0.948	0.961	0.905	0.946	0.959	0.894
		5	0.999	0.920	0.952	0.958	0.896	0.950	0.958	0.918
		10	1.000	0.906	0.951	0.955	0.898	0.953	0.955	0.937
		20	1.000	0.901	0.953	0.954	0.893	0.952	0.953	0.943
		40	1.000	0.900	0.955	0.956	0.898	0.953	0.953	0.947
		80	1.000	0.895	0.950	0.951	0.895	0.950	0.950	0.944
	11	3	0.999	0.935	0.949	0.961	0.908	0.945	0.957	0.892
		5	1.000	0.921	0.947	0.953	0.900	0.947	0.953	0.917
		10	1.000	0.914	0.953	0.956	0.902	0.951	0.954	0.934
		20	1.000	0.912	0.952	0.953	0.909	0.953	0.954	0.944
		40	1.000	0.910	0.954	0.955	0.910	0.955	0.956	0.950
		80	1.000	0.904	0.950	0.950	0.903	0.950	0.951	0.944
	20	3	0.999	0.940	0.947	0.954	0.925	0.947	0.956	0.894
		5	1.000	0.936	0.952	0.956	0.924	0.946	0.949	0.921
		10	1.000	0.932	0.952	0.955	0.925	0.948	0.950	0.935
		20	1.000	0.927	0.954	0.955	0.923	0.950	0.951	0.943
		40	1.000	0.923	0.949	0.949	0.925	0.948	0.949	0.945
		80	1.000	0.928	0.951	0.951	0.925	0.951	0.951	0.948

Table 2 continued: Actual coverage of nominal 95% confidence intervals

ρ	J	I	Incorrect Anova 1-way	Incorrect Anova 2-way	Corrected Anova (uses z)	Corrected Anova (uses t)	Incorrect Anocovar	Corrected Anocovar (uses z)	Corrected Anocovar (uses t)	MLE
.95	2	3	0.985	0.907	0.911	0.970	0.736	0.873	0.966	0.798
		5	0.987	0.821	0.929	0.961	0.648	0.914	0.955	0.852
		10	0.990	0.715	0.943	0.957	0.614	0.937	0.952	0.883
		20	0.992	0.644	0.947	0.954	0.595	0.945	0.952	0.897
		40	0.992	0.626	0.951	0.953	0.604	0.949	0.953	0.901
		80	0.992	0.606	0.948	0.949	0.594	0.948	0.949	0.902
	3	3	0.995	0.897	0.932	0.973	0.748	0.915	0.967	0.853
		5	0.996	0.838	0.941	0.962	0.699	0.933	0.956	0.887
		10	0.997	0.761	0.950	0.961	0.687	0.946	0.956	0.908
		20	0.998	0.706	0.954	0.958	0.668	0.951	0.956	0.918
		40	0.998	0.689	0.949	0.951	0.674	0.946	0.949	0.916
		80	0.999	0.676	0.948	0.949	0.668	0.948	0.949	0.919
	5	3	0.998	0.902	0.941	0.966	0.781	0.933	0.964	0.883
		5	0.999	0.866	0.948	0.961	0.774	0.949	0.960	0.913
		10	0.999	0.814	0.950	0.956	0.761	0.948	0.953	0.921
		20	1.000	0.776	0.950	0.953	0.752	0.947	0.950	0.927
		40	1.000	0.776	0.953	0.954	0.763	0.954	0.955	0.937
		80	1.000	0.756	0.950	0.951	0.751	0.948	0.949	0.932
	7	3	0.999	0.915	0.948	0.968	0.826	0.949	0.966	0.897
		5	1.000	0.881	0.950	0.960	0.809	0.947	0.957	0.914
		10	1.000	0.848	0.953	0.958	0.807	0.951	0.955	0.931
		20	1.000	0.827	0.951	0.953	0.804	0.951	0.954	0.934
		40	1.000	0.812	0.948	0.949	0.807	0.949	0.949	0.936
		80	1.000	0.804	0.950	0.950	0.800	0.950	0.951	0.938
	9	3	1.000	0.918	0.949	0.962	0.844	0.948	0.964	0.899
		5	1.000	0.896	0.953	0.960	0.847	0.950	0.958	0.922
		10	1.000	0.870	0.955	0.958	0.836	0.953	0.957	0.936
		20	1.000	0.853	0.952	0.953	0.832	0.950	0.951	0.941
		40	1.000	0.838	0.948	0.949	0.830	0.948	0.949	0.940
		80	1.000	0.837	0.952	0.953	0.832	0.952	0.953	0.943
	11	3	1.000	0.926	0.950	0.962	0.862	0.953	0.963	0.908
		5	1.000	0.912	0.954	0.960	0.853	0.950	0.957	0.924
		10	1.000	0.877	0.950	0.953	0.849	0.951	0.953	0.934
		20	1.000	0.868	0.954	0.955	0.855	0.951	0.952	0.942
		40	1.000	0.857	0.947	0.947	0.851	0.947	0.948	0.940
		80	1.000	0.850	0.949	0.949	0.849	0.951	0.951	0.946
	20	3	1.000	0.938	0.952	0.958	0.895	0.949	0.956	0.903
		5	1.000	0.927	0.953	0.957	0.897	0.952	0.956	0.926
		10	1.000	0.914	0.954	0.955	0.899	0.953	0.955	0.941
		20	1.000	0.902	0.950	0.951	0.897	0.950	0.951	0.943
		40	1.000	0.897	0.949	0.949	0.897	0.948	0.948	0.943
		80	1.000	0.901	0.951	0.951	0.899	0.952	0.952	0.949

Table 2 continued: Actual coverage of nominal 95% confidence intervals

			Incorrect Anova		Corrected Anova			Corrected Anocovar		
ρ	J	I	1-way	2-way	(uses z)	(uses t)	Incorrect Anocovar	(uses z)	(uses t)	MLE
.99	2	3	0.989	0.890	0.915	0.974	0.433	0.869	0.964	0.799
		5	0.991	0.753	0.932	0.964	0.336	0.914	0.955	0.839
		10	0.992	0.555	0.941	0.955	0.306	0.933	0.950	0.864
		20	0.992	0.433	0.948	0.955	0.308	0.947	0.954	0.885
		40	0.994	0.367	0.950	0.955	0.305	0.948	0.953	0.890
		80	0.994	0.336	0.952	0.954	0.309	0.951	0.953	0.895
	3	3	0.997	0.885	0.940	0.977	0.439	0.918	0.969	0.858
		5	0.998	0.765	0.948	0.967	0.392	0.933	0.959	0.879
		10	0.999	0.610	0.952	0.962	0.376	0.945	0.955	0.899
		20	0.999	0.487	0.949	0.954	0.372	0.948	0.952	0.906
		40	1.000	0.428	0.949	0.952	0.363	0.948	0.951	0.904
		80	0.999	0.392	0.950	0.950	0.365	0.949	0.950	0.909
	5	3	1.000	0.890	0.948	0.972	0.491	0.940	0.966	0.894
		5	1.000	0.804	0.949	0.963	0.465	0.948	0.960	0.911
		10	1.000	0.689	0.952	0.958	0.461	0.948	0.954	0.918
		20	1.000	0.573	0.947	0.950	0.448	0.945	0.947	0.918
		40	1.000	0.511	0.951	0.953	0.452	0.953	0.955	0.930
		80	1.000	0.475	0.949	0.949	0.440	0.948	0.949	0.927
	7	3	1.000	0.895	0.948	0.966	0.537	0.949	0.967	0.908
		5	1.000	0.843	0.955	0.963	0.519	0.952	0.961	0.924
		10	1.000	0.740	0.956	0.959	0.518	0.952	0.954	0.930
		20	1.000	0.634	0.954	0.956	0.514	0.952	0.953	0.933
		40	1.000	0.577	0.953	0.954	0.518	0.953	0.954	0.938
		80	1.000	0.543	0.953	0.954	0.515	0.953	0.954	0.939
	9	3	1.000	0.911	0.951	0.965	0.589	0.955	0.968	0.919
		5	1.000	0.859	0.956	0.963	0.569	0.952	0.959	0.922
		10	1.000	0.770	0.954	0.957	0.564	0.954	0.957	0.934
		20	1.000	0.680	0.955	0.957	0.560	0.953	0.955	0.938
		40	1.000	0.626	0.951	0.952	0.561	0.950	0.951	0.937
		80	1.000	0.594	0.951	0.951	0.563	0.952	0.952	0.940
	11	3	1.000	0.919	0.954	0.965	0.612	0.954	0.966	0.919
		5	1.000	0.874	0.953	0.959	0.607	0.953	0.960	0.929
		10	1.000	0.795	0.957	0.959	0.594	0.951	0.954	0.932
		20	1.000	0.713	0.954	0.955	0.595	0.952	0.953	0.940
		40	1.000	0.656	0.951	0.951	0.599	0.951	0.951	0.941
		80	1.000	0.629	0.952	0.952	0.603	0.951	0.951	0.942
	20	3	1.000	0.930	0.949	0.955	0.716	0.957	0.963	0.931
		5	1.000	0.905	0.951	0.954	0.717	0.955	0.958	0.934
		10	1.000	0.862	0.954	0.957	0.708	0.953	0.955	0.939
		20	1.000	0.802	0.951	0.951	0.710	0.949	0.950	0.940
		40	1.000	0.764	0.953	0.954	0.715	0.954	0.954	0.946
		80	1.000	0.725	0.951	0.951	0.700	0.949	0.949	0.944

Table 2 continued: Actual coverage of nominal 95% confidence intervals

ρ	J	Corrected Anova		Corrected Anocov	
		uses z	uses t	uses z	uses t
.5	2	18	3	17	3
	3	10	3	9	3
	5	6	3	6	3
	7	5	3	5	3
	9	4	3	4	3
	11	3	3	4	3
	20	3	3	3	3
.6	2	14	3	15	4
	3	8	3	9	3
	5	6	3	6	3
	7	5	3	5	3
	9	5	3	5	3
	11	4	3	4	3
	20	3	3	3	3
.7	2	10	3	11	5
	3	7	3	8	4
	5	6	3	7	3
	7	4	3	4	3
	9	3	3	4	3
	11	4	3	3	3
	20	3	3	3	3
.8	2	8	8	10	5
	3	6	5	7	5
	5	4	3	5	3
	7	3	3	4	3
	9	3	3	3	3
	11	3	3	3	3
	20	3	3	3	3

Table 3: I's needed to yield actual coverages that lie between 0.94 and 0.96 for nominal 95% confidence intervals.

ρ	J	Corrected Anova uses z	Corrected Anova uses t	Corrected Anocov uses z	Corrected Anocov uses t
.9	2	8	5	13	4
	3	6	6	7	5
	5	3	5	4	4
	7	3	3	4	3
	9	3	4	3	3
	11	3	4	3	3
	20	3	3	3	3
.95	2	9	7	12	4
	3	5	9	7	5
	5	3	6	4	5
	7	3	6	3	5
	9	3	6	3	5
	11	3	5	3	4
	20	3	3	3	3
.99	2	9	7	13	4
	3	3	10	7	5
	5	3	7	3	5
	7	3	9	3	5
	9	3	8	3	6
	11	3	6	3	5
	20	3	3	3	4

Table 3 continued: I's needed to yield actual coverages that lie between 0.94 and 0.96 for nominal 95% confidence intervals.

ρ	J	I	1-way $\hat{s}$			2-way $\hat{s}$
			no ρ	$\hat{\rho}$	ρ_{true}	
0.00	3	3	0.051	0.089	0.051	0.050
		5	0.051	0.065	0.051	0.051
		10	0.049	0.055	0.049	0.050
		20	0.051	0.054	0.051	0.051
		40	0.050	0.051	0.050	0.050
	5	3	0.050	0.072	0.050	0.050
		5	0.050	0.060	0.050	0.050
		10	0.050	0.054	0.050	0.049
		20	0.049	0.051	0.049	0.050
		40	0.050	0.051	0.050	0.050
	7	3	0.049	0.066	0.049	0.049
		5	0.050	0.058	0.050	0.050
		10	0.050	0.053	0.050	0.049
		20	0.050	0.051	0.050	0.050
		40	0.050	0.050	0.050	0.049
	9	3	0.050	0.064	0.050	0.050
		5	0.050	0.057	0.050	0.051
		10	0.051	0.054	0.051	0.051
		20	0.051	0.053	0.051	0.051
		40	0.051	0.051	0.051	0.050
	11	3	0.050	0.061	0.050	0.050
		5	0.050	0.056	0.050	0.050
		10	0.050	0.053	0.050	0.051
		20	0.050	0.051	0.050	0.049
		40	0.050	0.050	0.050	0.050
	20	3	0.050	0.057	0.050	0.050
		5	0.051	0.055	0.051	0.051
		10	0.049	0.051	0.049	0.049
		20	0.050	0.051	0.050	0.050
		40	0.050	0.051	0.050	0.050

Table 4: Tukey test size simulation (see Section 5.3 for details), $\rho = 0.0$

ρ	J	I	1-way $\hat{s}$ no ρ	1-way $\hat{s}$ $\hat{\rho}$	1-way $\hat{s}$ ρ_{true}	2-way $\hat{s}$
0.50	3	3	0.028	0.110	0.050	0.051
		5	0.023	0.068	0.049	0.048
		10	0.022	0.059	0.051	0.050
		20	0.020	0.054	0.051	0.050
		40	0.019	0.051	0.049	0.050
	5	3	0.023	0.093	0.049	0.051
		5	0.018	0.065	0.049	0.050
		10	0.016	0.058	0.050	0.052
		20	0.015	0.053	0.050	0.050
		40	0.014	0.050	0.049	0.049
	7	3	0.018	0.080	0.047	0.050
		5	0.015	0.062	0.047	0.050
		10	0.013	0.055	0.048	0.050
		20	0.013	0.053	0.050	0.050
		40	0.012	0.051	0.049	0.050
	9	3	0.015	0.075	0.046	0.050
		5	0.013	0.060	0.047	0.050
		10	0.011	0.054	0.047	0.050
		20	0.011	0.052	0.048	0.050
		40	0.011	0.052	0.050	0.051
	11	3	0.014	0.071	0.046	0.050
		5	0.011	0.059	0.047	0.050
		10	0.010	0.053	0.047	0.049
		20	0.010	0.052	0.049	0.050
		40	0.009	0.051	0.049	0.050
	20	3	0.009	0.060	0.044	0.050
		5	0.007	0.057	0.046	0.051
		10	0.007	0.051	0.045	0.048
		20	0.006	0.051	0.048	0.050
		40	0.006	0.051	0.049	0.050

Table 4 continued: Tukey test size simulation (see Section 5.3 for details), $\rho = 0.50$

ρ	J	I	1-way $\hat{s}$ no ρ	1-way $\hat{s}$ $\hat{\rho}$	1-way $\hat{s}$ ρ_{true}	2-way $\hat{s}$
0.60	3	3	0.021	0.125	0.051	0.050
		5	0.015	0.077	0.050	0.050
		10	0.012	0.060	0.050	0.049
		20	0.011	0.054	0.049	0.049
		40	0.010	0.053	0.050	0.051
	5	3	0.013	0.108	0.049	0.049
		5	0.010	0.072	0.049	0.049
		10	0.007	0.060	0.049	0.050
		20	0.006	0.054	0.050	0.050
		40	0.006	0.052	0.049	0.050
	7	3	0.009	0.097	0.047	0.049
		5	0.007	0.070	0.048	0.050
		10	0.005	0.057	0.047	0.049
		20	0.004	0.053	0.047	0.050
		40	0.005	0.052	0.050	0.051
	9	3	0.008	0.090	0.047	0.050
		5	0.005	0.067	0.046	0.049
		10	0.004	0.058	0.048	0.051
		20	0.004	0.054	0.050	0.051
		40	0.003	0.051	0.049	0.049
	11	3	0.006	0.084	0.046	0.050
		5	0.004	0.066	0.046	0.049
		10	0.003	0.056	0.047	0.049
		20	0.004	0.053	0.048	0.050
		40	0.003	0.052	0.049	0.051
	20	3	0.003	0.073	0.045	0.051
		5	0.002	0.063	0.045	0.050
		10	0.001	0.057	0.046	0.050
		20	0.002	0.053	0.048	0.050
		40	0.002	0.052	0.049	0.050

Table 4 continued: Tukey test size simulation (see Section 5.3 for details), $\rho = 0.60$

ρ	J	I	1-way $\hat{s}$			2-way $\hat{s}$
			no ρ	$\hat{\rho}$	ρ_{true}	
0.70	3	3	0.012	0.147	0.053	0.050
		5	0.007	0.085	0.052	0.049
		10	0.005	0.065	0.052	0.050
		20	0.004	0.056	0.050	0.051
		40	0.003	0.051	0.049	0.048
	5	3	0.006	0.136	0.051	0.049
		5	0.004	0.084	0.051	0.050
		10	0.002	0.065	0.051	0.050
		20	0.002	0.057	0.051	0.052
		40	0.001	0.052	0.049	0.049
	7	3	0.004	0.126	0.050	0.050
		5	0.002	0.083	0.050	0.049
		10	0.001	0.064	0.050	0.050
		20	0.001	0.055	0.049	0.049
		40	0.001	0.052	0.048	0.049
	9	3	0.003	0.122	0.050	0.050
		5	0.001	0.083	0.050	0.051
		10	0.001	0.062	0.047	0.049
		20	0.001	0.056	0.048	0.049
		40	0.001	0.052	0.048	0.049
	11	3	0.002	0.117	0.050	0.049
		5	0.001	0.081	0.049	0.049
		10	0.001	0.062	0.048	0.049
		20	0.000	0.056	0.049	0.050
		40	0.000	0.053	0.049	0.050
	20	3	0.000	0.107	0.051	0.052
		5	0.000	0.081	0.049	0.052
		10	0.000	0.064	0.048	0.050
		20	0.000	0.055	0.048	0.049
		40	0.000	0.054	0.049	0.051

Table 4 continued: Tukey test size simulation (see Section 5.3 for details), $\rho = 0.70$

ρ	J	I	1-way $\hat{s}$			2-way $\hat{s}$
			no ρ	$\hat{\rho}$	ρ_{true}	
0.80	3	3	0.005	0.191	0.060	0.049
		5	0.002	0.107	0.061	0.050
		10	0.001	0.073	0.056	0.050
		20	0.001	0.060	0.052	0.050
		40	0.000	0.055	0.051	0.051
	5	3	0.002	0.193	0.063	0.049
		5	0.001	0.112	0.058	0.049
		10	0.000	0.075	0.054	0.051
		20	0.000	0.061	0.052	0.051
		40	0.000	0.054	0.050	0.050
	7	3	0.001	0.193	0.061	0.049
		5	0.000	0.114	0.060	0.050
		10	0.000	0.076	0.053	0.050
		20	0.000	0.060	0.050	0.049
		40	0.000	0.055	0.049	0.050
	9	3	0.000	0.194	0.064	0.049
		5	0.000	0.117	0.059	0.051
		10	0.000	0.076	0.053	0.051
		20	0.000	0.061	0.050	0.049
		40	0.000	0.055	0.050	0.050
	11	3	0.000	0.192	0.064	0.050
		5	0.000	0.119	0.059	0.051
		10	0.000	0.076	0.052	0.049
		20	0.000	0.063	0.050	0.050
		40	0.000	0.055	0.049	0.050
	20	3	0.000	0.196	0.073	0.052
		5	0.000	0.127	0.060	0.050
		10	0.000	0.080	0.053	0.050
		20	0.000	0.063	0.050	0.050
		40	0.000	0.057	0.050	0.051

Table 4 continued: Tukey test size simulation (see Section 5.3 for details), $\rho = 0.80$

ρ	J	I	1-way $\hat{s}$ no ρ	1-way $\hat{s}$ $\hat{\rho}$	1-way $\hat{s}$ ρ_{true}	2-way $\hat{s}$
0.90	3	3	0.001	0.288	0.095	0.047
		5	0.000	0.161	0.087	0.048
		10	0.000	0.094	0.067	0.049
		20	0.000	0.069	0.058	0.050
		40	0.000	0.058	0.053	0.049
	5	3	0.000	0.329	0.112	0.047
		5	0.000	0.187	0.095	0.048
		10	0.000	0.104	0.069	0.049
		20	0.000	0.074	0.059	0.051
		40	0.000	0.060	0.053	0.050
	7	3	0.000	0.360	0.129	0.048
		5	0.000	0.209	0.102	0.050
		10	0.000	0.112	0.071	0.050
		20	0.000	0.075	0.059	0.051
		40	0.000	0.060	0.053	0.050
	9	3	0.000	0.393	0.145	0.050
		5	0.000	0.226	0.105	0.049
		10	0.000	0.118	0.073	0.051
		20	0.000	0.076	0.059	0.050
		40	0.000	0.062	0.053	0.050
	11	3	0.000	0.410	0.157	0.048
		5	0.000	0.243	0.114	0.050
		10	0.000	0.121	0.073	0.050
		20	0.000	0.078	0.059	0.050
		40	0.000	0.062	0.053	0.050
	20	3	0.000	0.481	0.209	0.052
		5	0.000	0.297	0.132	0.052
		10	0.000	0.139	0.079	0.051
		20	0.000	0.083	0.060	0.050
		40	0.000	0.063	0.052	0.050

Table 4 continued: Tukey test size simulation (see Section 5.3 for details), $\rho = 0.90$

ρ	J	I	1-way $\hat{s}$			2-way $\hat{s}$
			no ρ	$\hat{\rho}$	ρ_{true}	
0.95	3	3	0.000	0.420	0.191	0.045
		5	0.000	0.259	0.158	0.046
		10	0.000	0.140	0.102	0.048
		20	0.000	0.087	0.071	0.049
		40	0.000	0.065	0.059	0.050
	5	3	0.000	0.520	0.263	0.045
		5	0.000	0.329	0.193	0.046
		10	0.000	0.166	0.113	0.048
		20	0.000	0.096	0.075	0.050
		40	0.000	0.067	0.059	0.048
	7	3	0.000	0.585	0.323	0.045
		5	0.000	0.382	0.222	0.046
		10	0.000	0.185	0.122	0.048
		20	0.000	0.102	0.077	0.049
		40	0.000	0.070	0.059	0.049
	9	3	0.000	0.646	0.379	0.046
		5	0.000	0.430	0.249	0.047
		10	0.000	0.203	0.128	0.049
		20	0.000	0.106	0.080	0.050
		40	0.000	0.072	0.061	0.050
	11	3	0.000	0.685	0.424	0.046
		5	0.000	0.468	0.274	0.047
		10	0.000	0.217	0.136	0.049
		20	0.000	0.112	0.081	0.049
		40	0.000	0.072	0.061	0.050
	20	3	0.000	0.816	0.585	0.053
		5	0.000	0.603	0.358	0.053
		10	0.000	0.268	0.161	0.051
		20	0.000	0.124	0.087	0.050
		40	0.000	0.079	0.064	0.051

Table 4 continued: Tukey test size simulation (see Section 5.3 for details), $\rho = 0.95$

ρ	J	I	1-way $\hat{s}$			2-way $\hat{s}$
			no ρ	$\hat{\rho}$	ρ_{true}	
0.99	3	3	0.000	0.751	0.654	0.037
		5	0.000	0.631	0.556	0.037
		10	0.000	0.415	0.363	0.041
		20	0.000	0.230	0.200	0.045
		40	0.000	0.126	0.114	0.049
	5	3	0.000	0.889	0.839	0.038
		5	0.000	0.801	0.729	0.038
		10	0.000	0.554	0.481	0.040
		20	0.000	0.291	0.250	0.044
		40	0.000	0.147	0.132	0.049
	7	3	0.000	0.946	0.922	0.038
		5	0.000	0.888	0.827	0.037
		10	0.000	0.645	0.563	0.043
		20	0.000	0.343	0.294	0.046
		40	0.000	0.163	0.143	0.048
	9	3	0.000	0.976	0.963	0.039
		5	0.000	0.937	0.891	0.041
		10	0.000	0.717	0.635	0.045
		20	0.000	0.384	0.326	0.047
		40	0.000	0.177	0.154	0.049
	11	3	0.000	0.987	0.981	0.040
		5	0.000	0.962	0.926	0.042
		10	0.000	0.766	0.683	0.045
		20	0.000	0.416	0.352	0.048
		40	0.000	0.188	0.163	0.049
	20	3	0.000	0.999	0.999	0.052
		5	0.000	0.996	0.988	0.056
		10	0.000	0.902	0.836	0.058
		20	0.000	0.539	0.456	0.055
		40	0.000	0.231	0.196	0.053

Table 4 continued: Tukey test size simulation (see Section 5.3 for details), $\rho = 0.99$

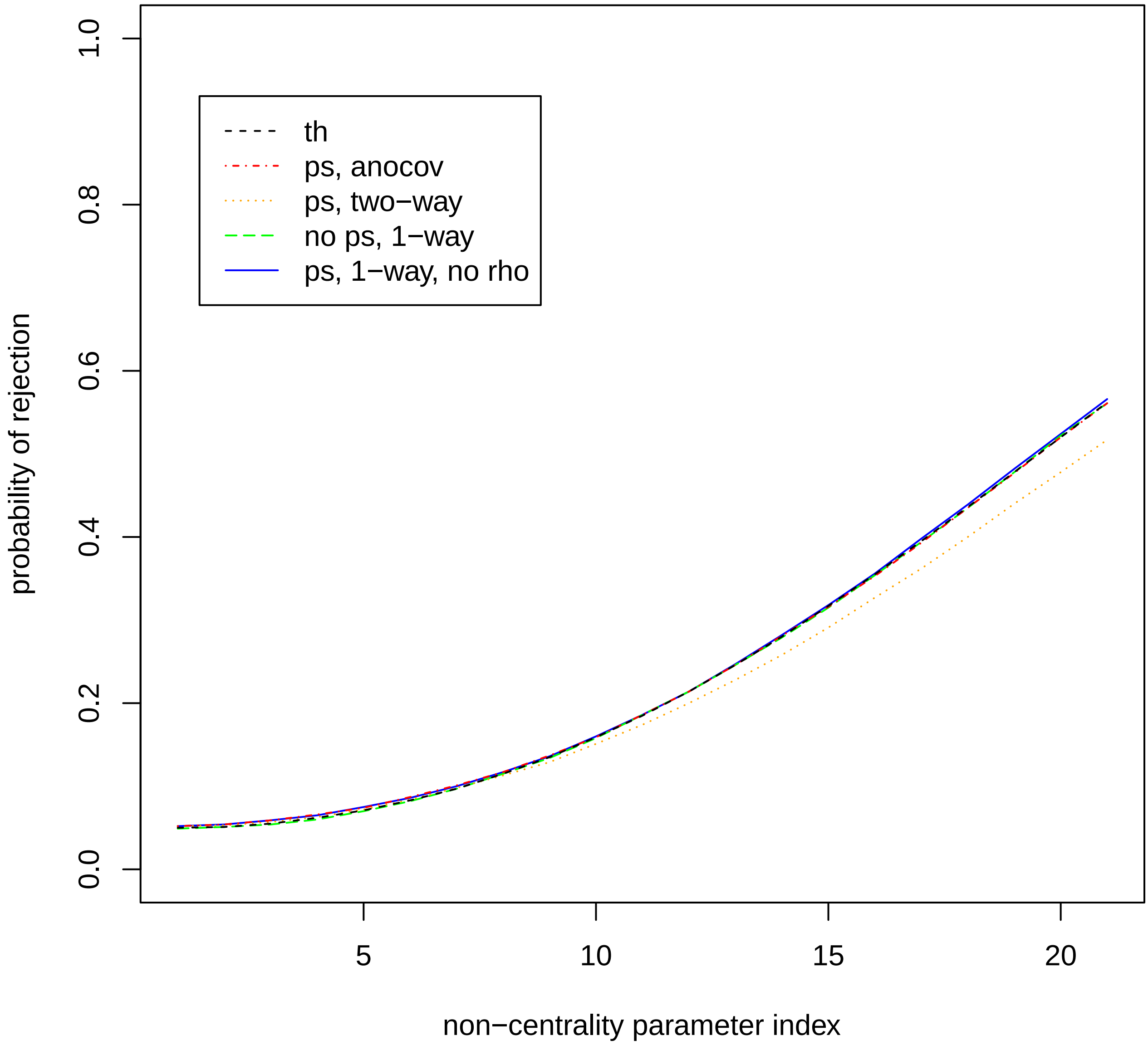

Figure 1: Power plot, $\rho = 0.0$, $J = 2$, $I = 10$.

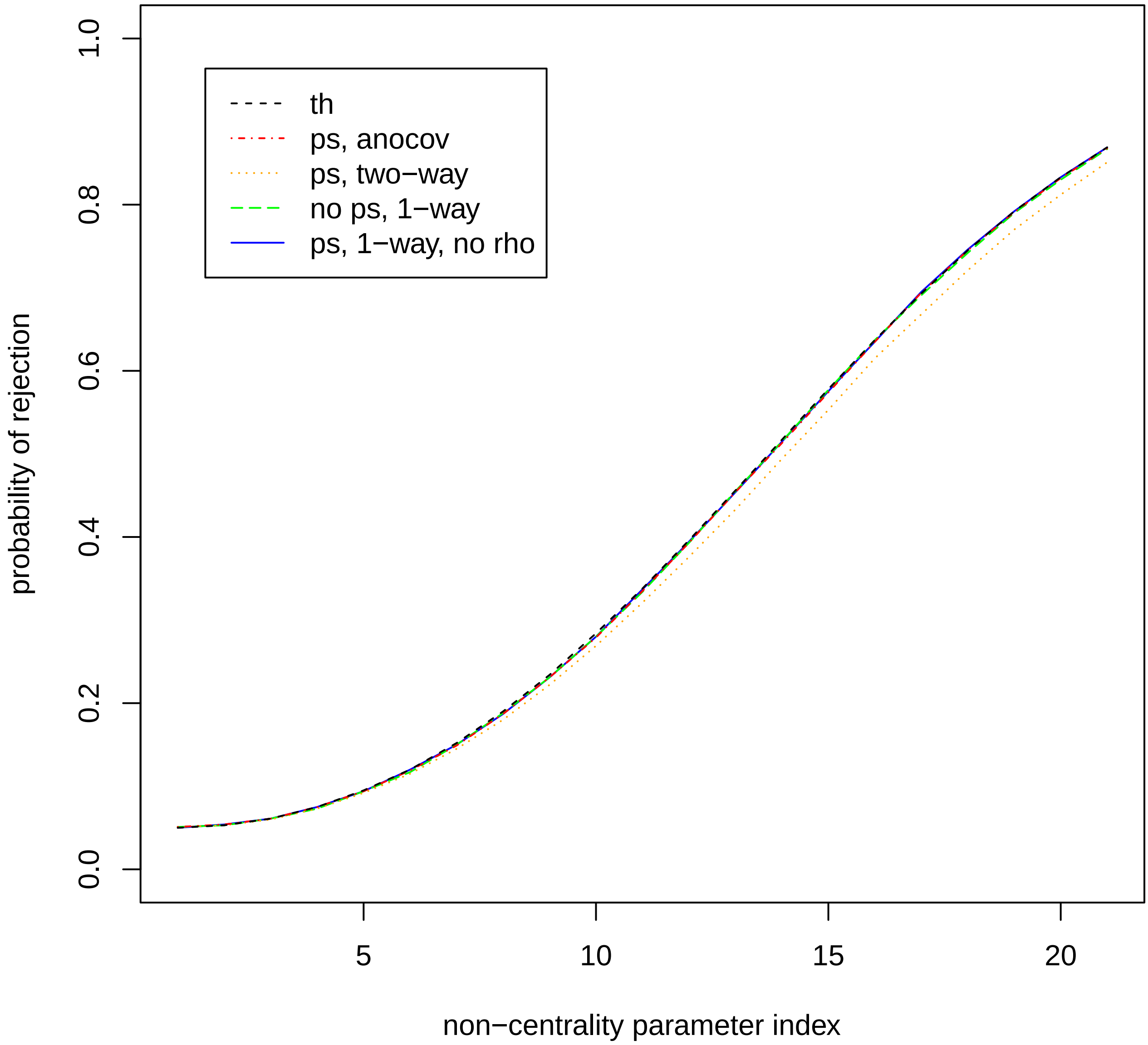

Figure 2: Power plot, $\rho = 0.0$, $J = 2$, $I = 20$.

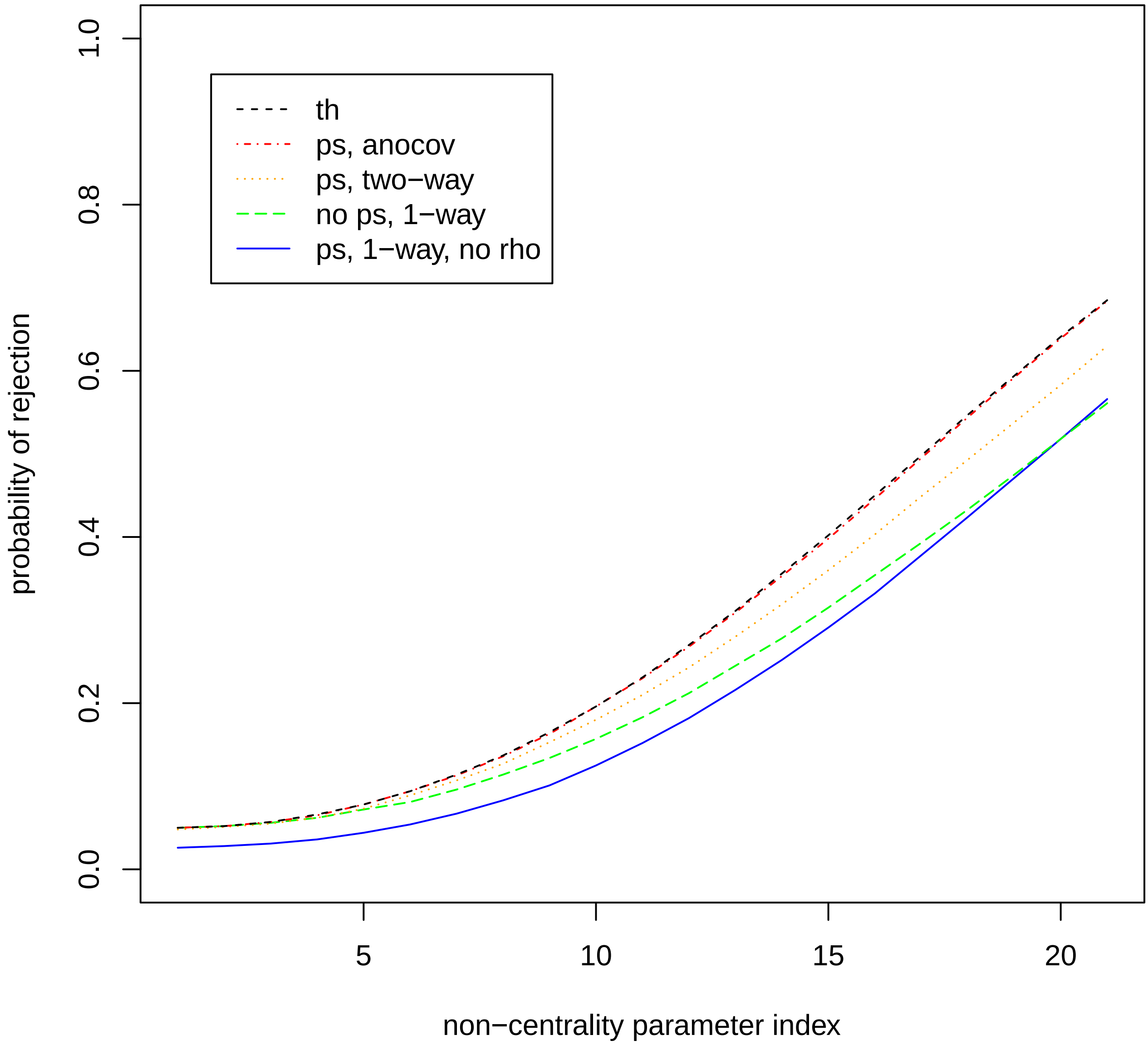

Figure 3: Power plot, $\rho = 0.5$, $J = 2$, $I = 10$.

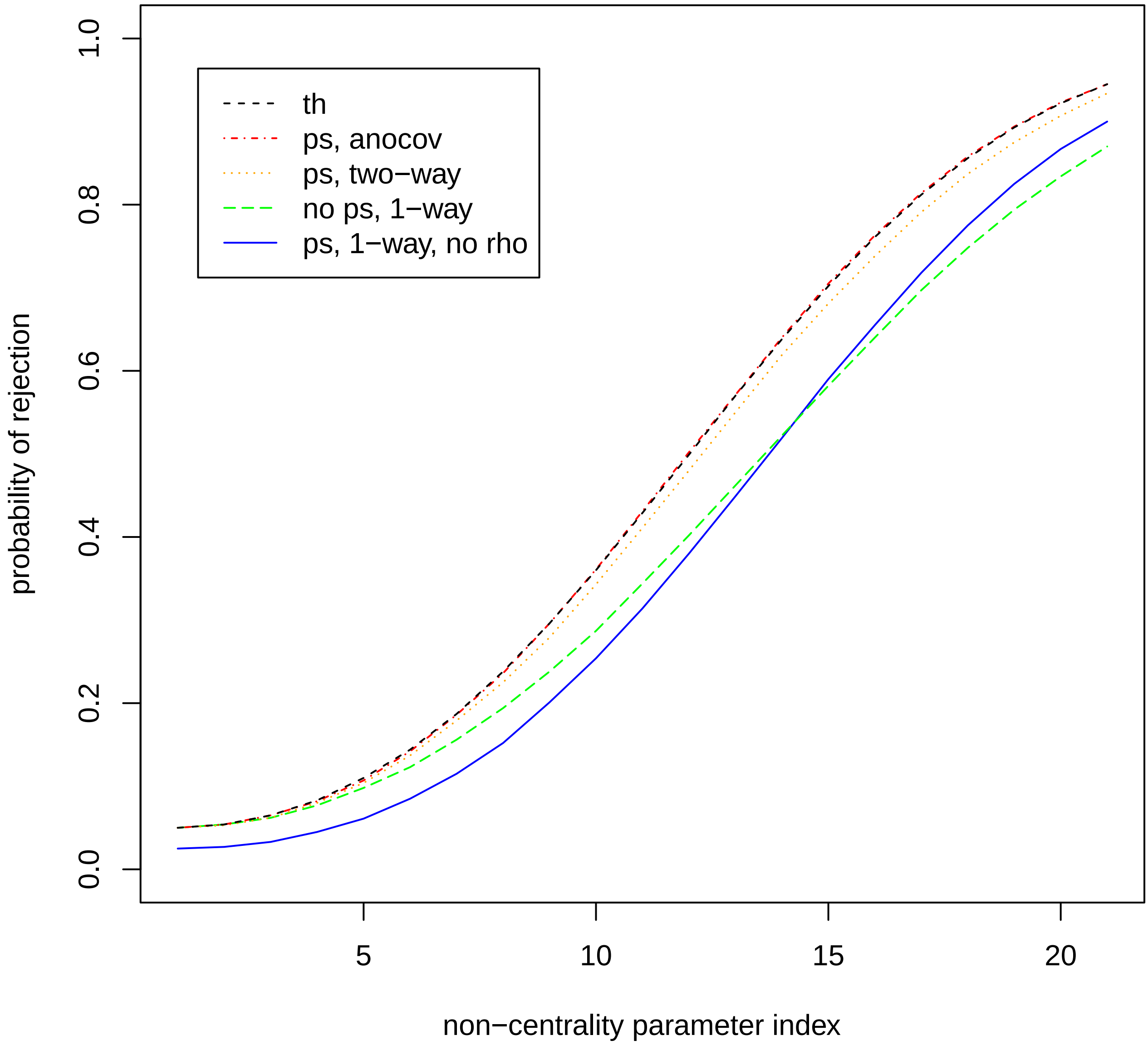

Figure 4: Power plot, $\rho = 0.5$, $J = 2$, $I = 20$.

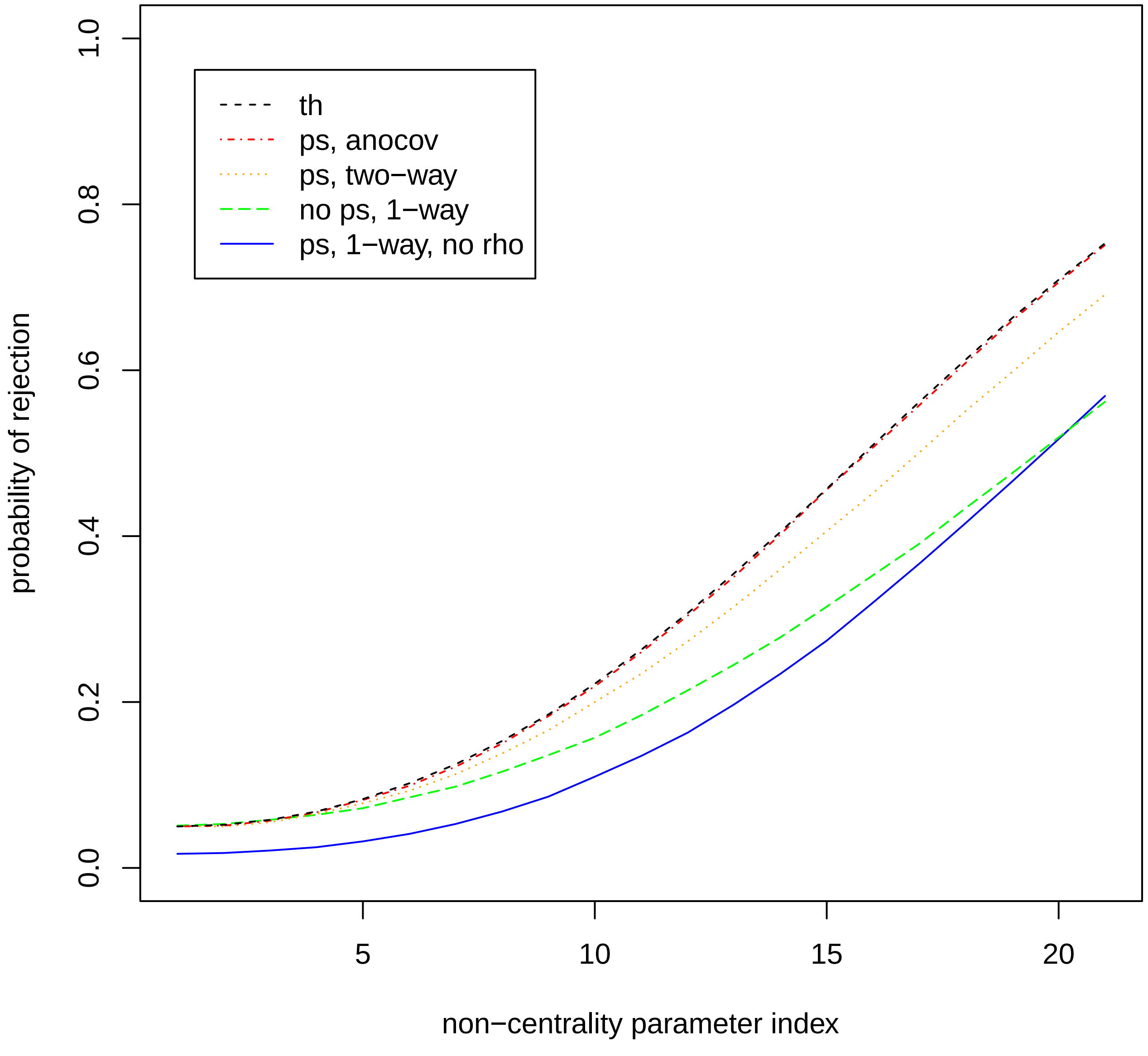

Figure 5: Power plot, $\rho = 0.6$, $J = 2$, $I = 10$.

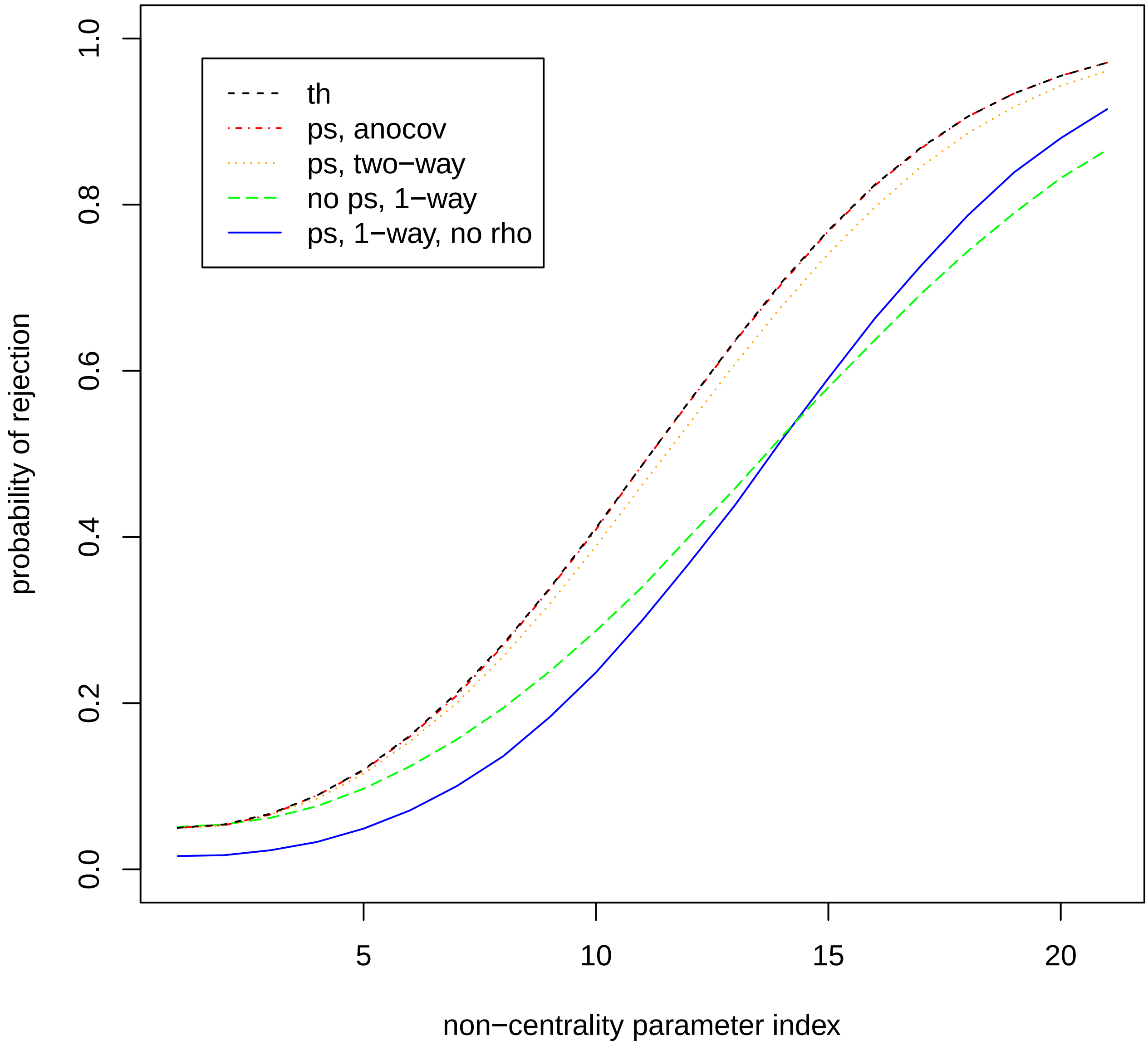

Figure 6: Power plot, $\rho = 0.6$, $J = 2$, $I = 20$.

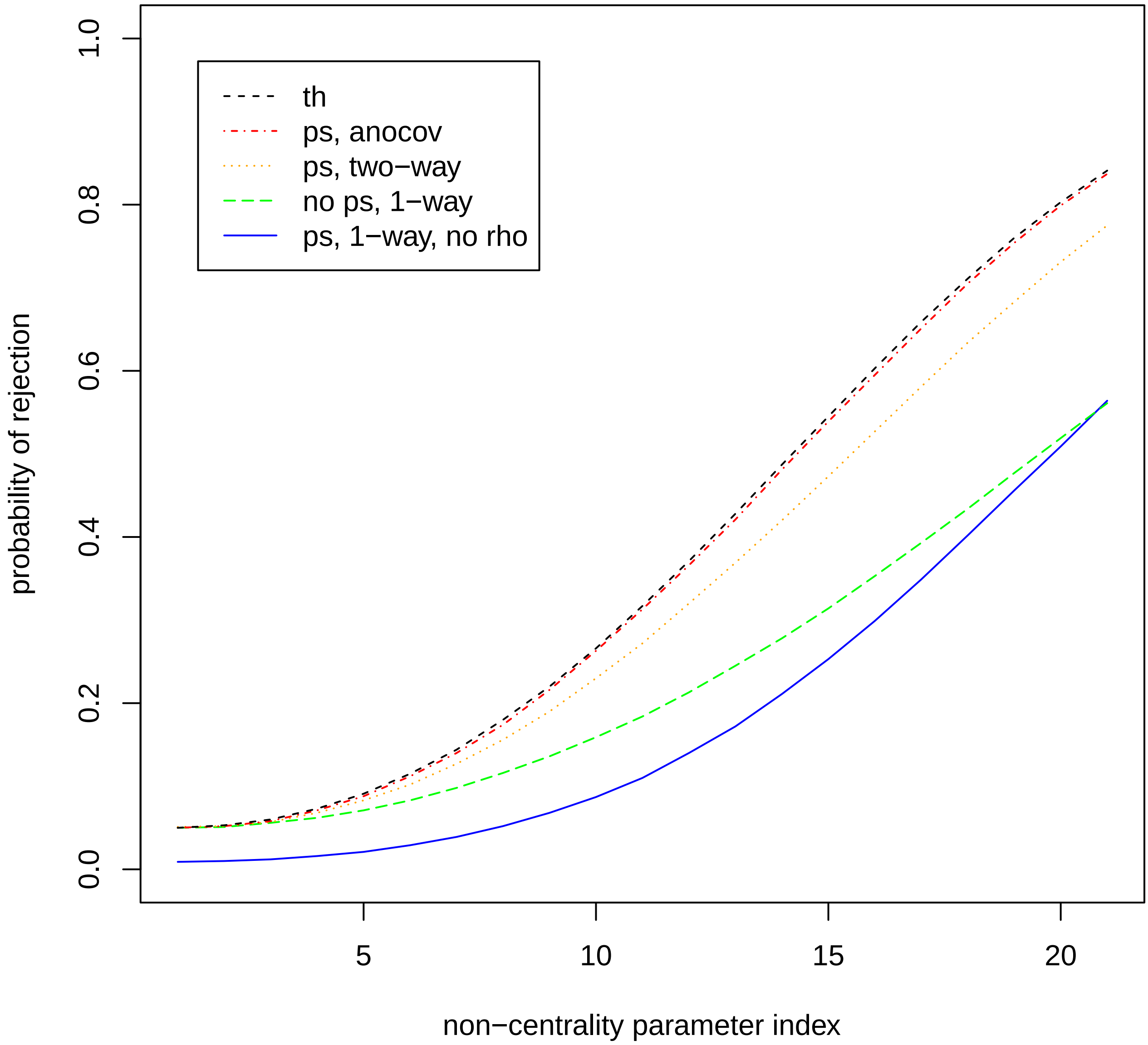

Figure 7: Power plot, $\rho = 0.7$, $J = 2$, $I = 10$.

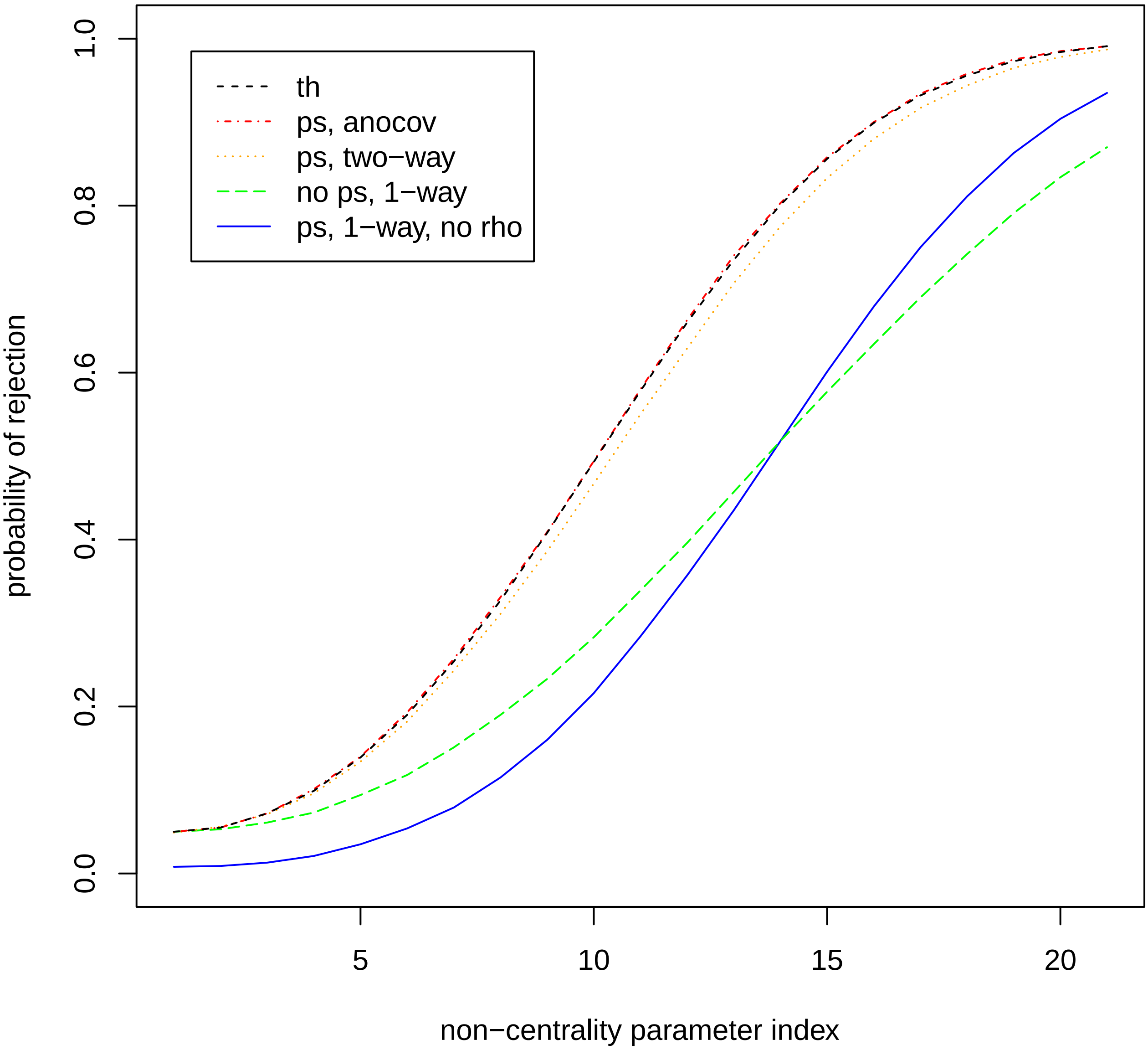

Figure 8: Power plot, $\rho = 0.7$, $J = 2$, $I = 20$.

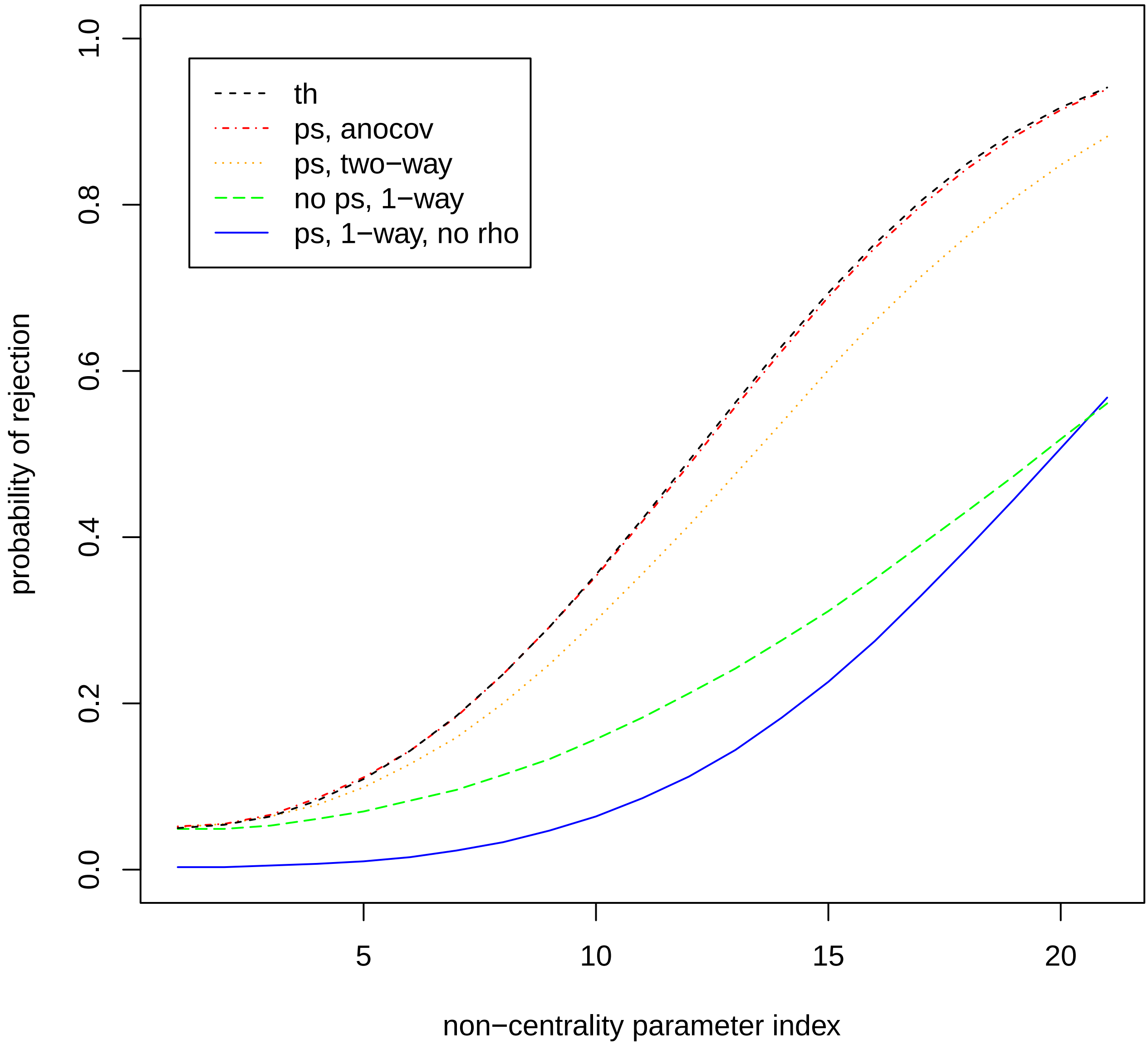

Figure 9: Power plot, $\rho = 0.8$, $J = 2$, $I = 10$.

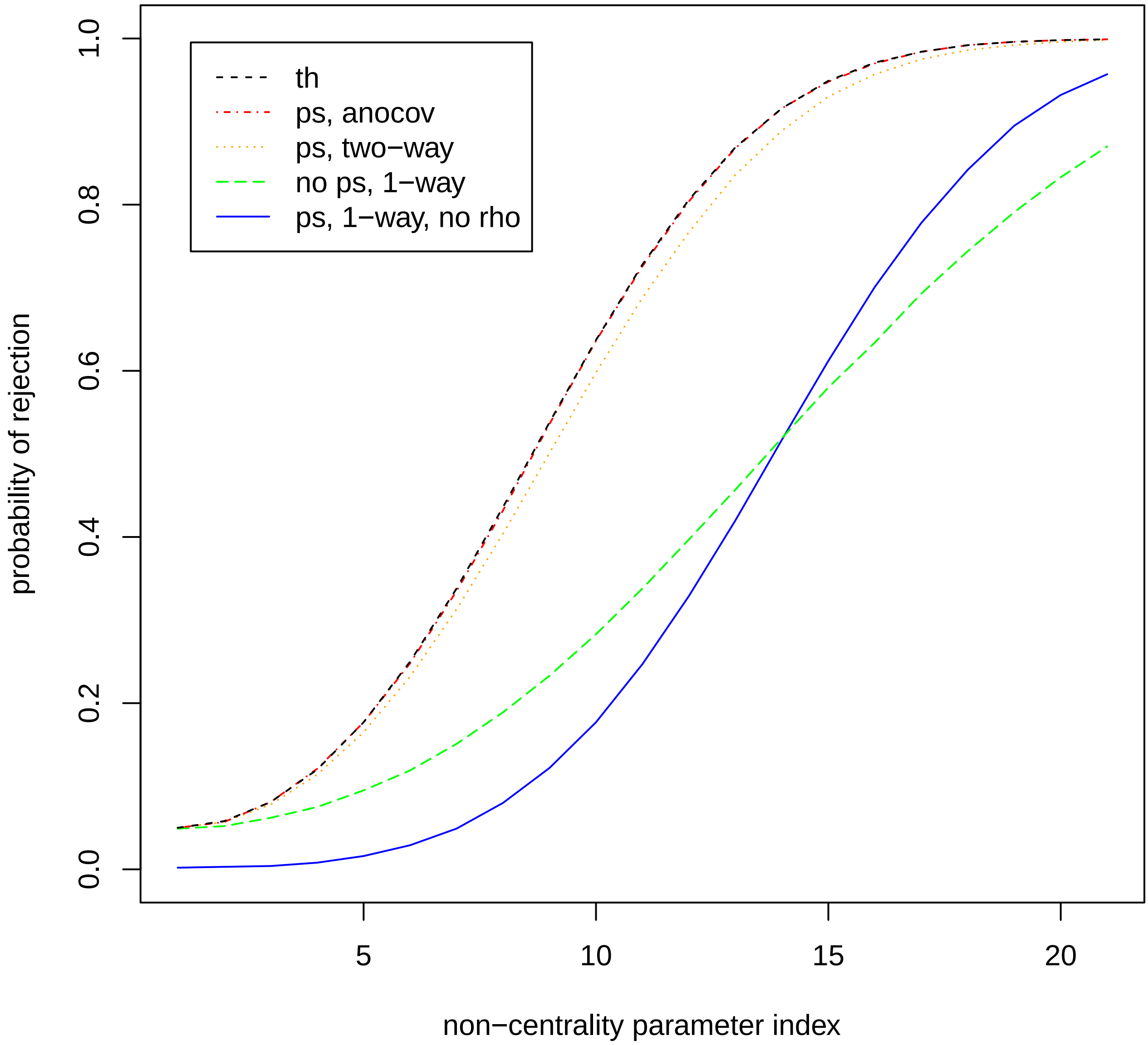

Figure 10: Power plot, $\rho = 0.8$, $J = 2$, $I = 20$.

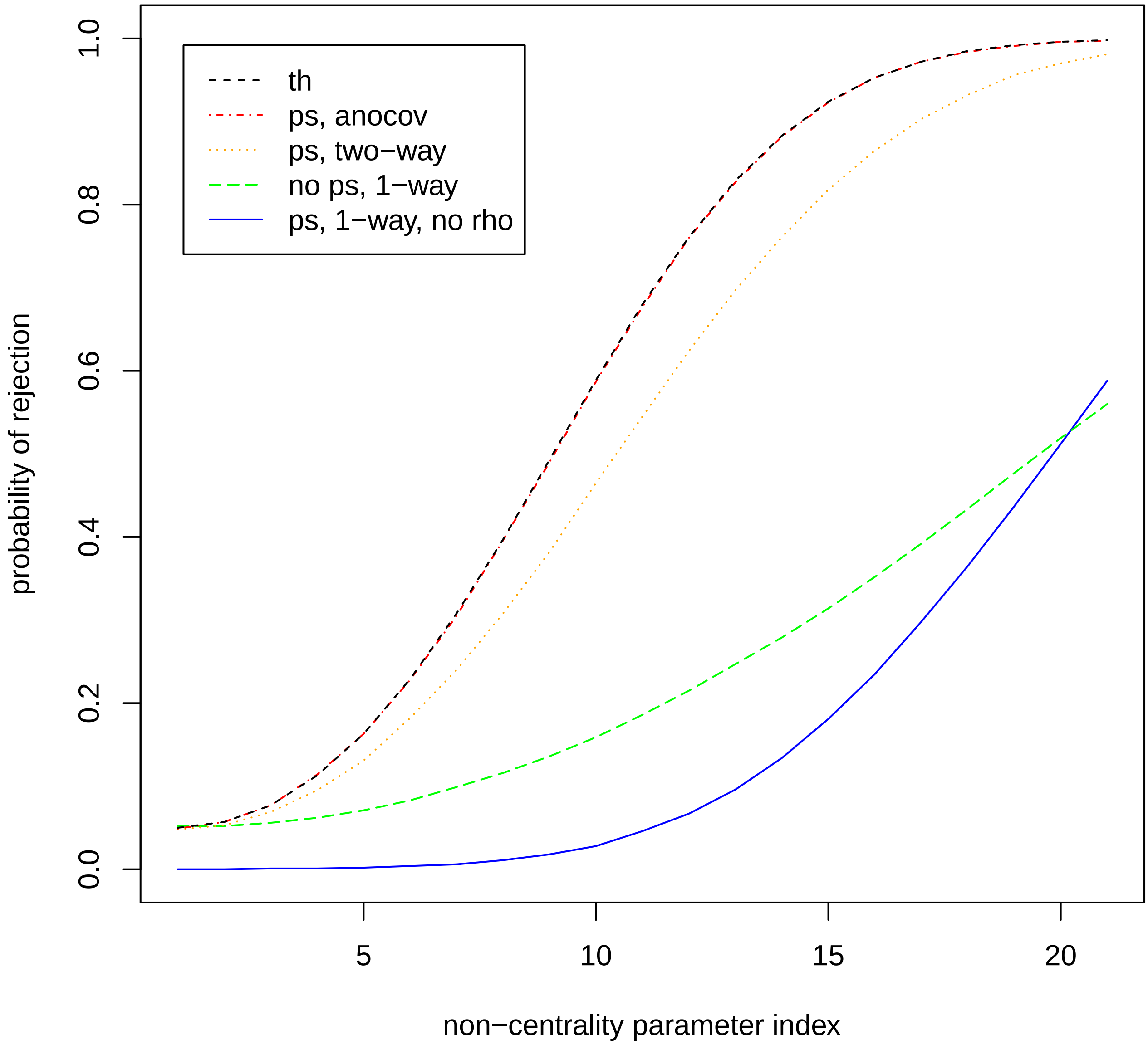

Figure 11: Power plot, $\rho = 0.9$, $J = 2$, $I = 10$.

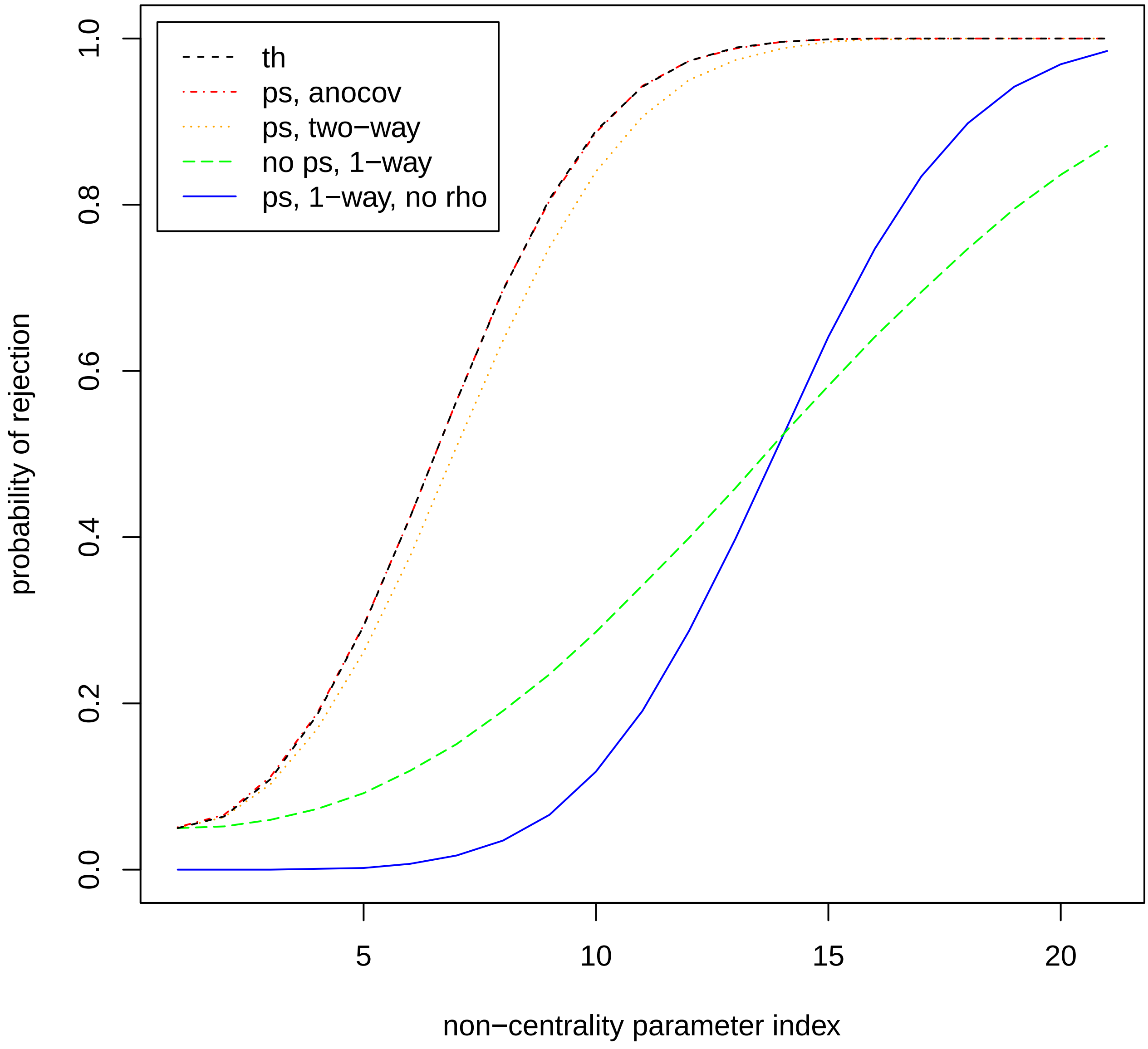

Figure 12: Power plot, $\rho = 0.9$, $J = 2$, $I = 20$.

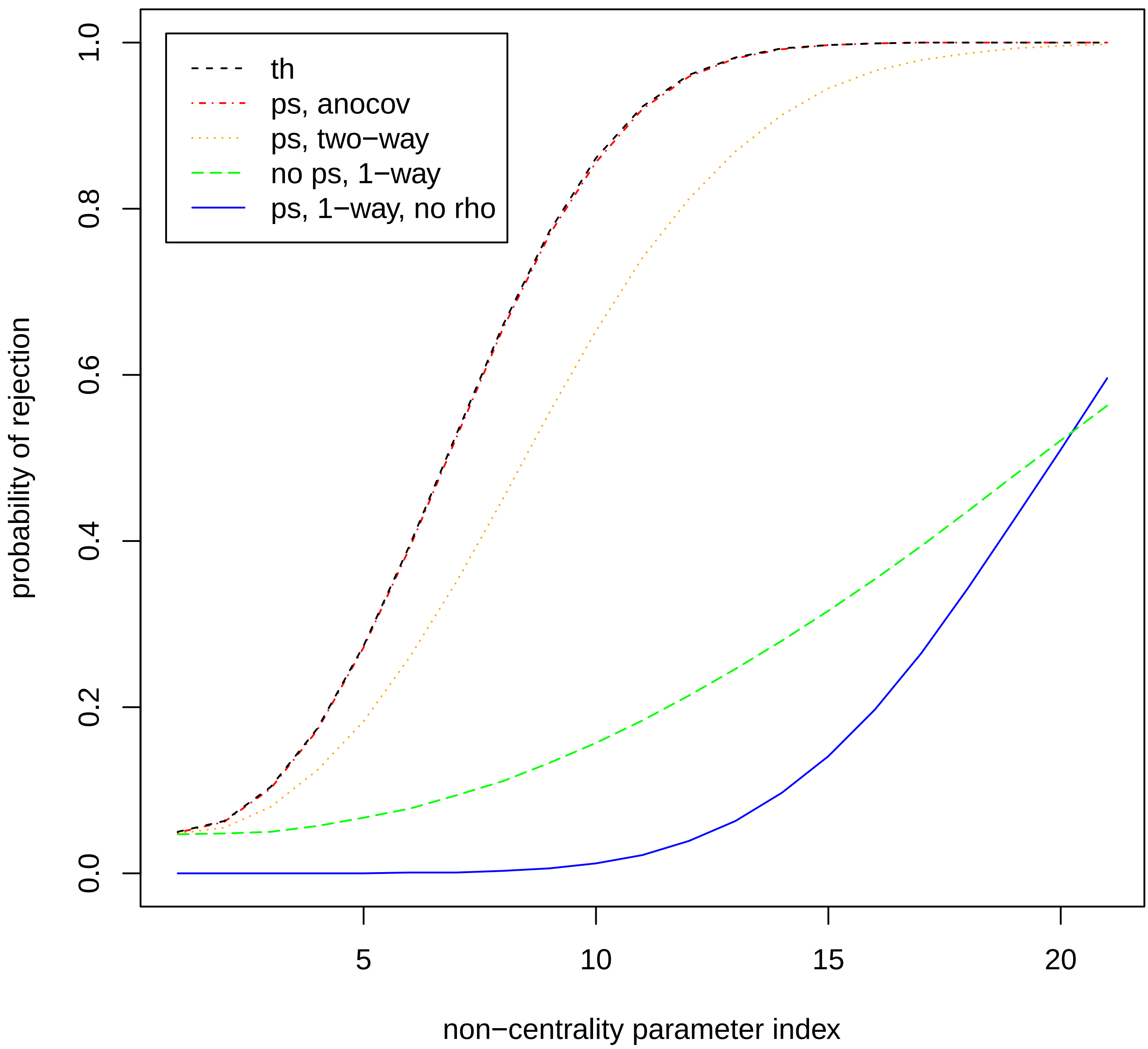

Figure 13: Power plot, $\rho = 0.95$, $J = 2$, $I = 10$.

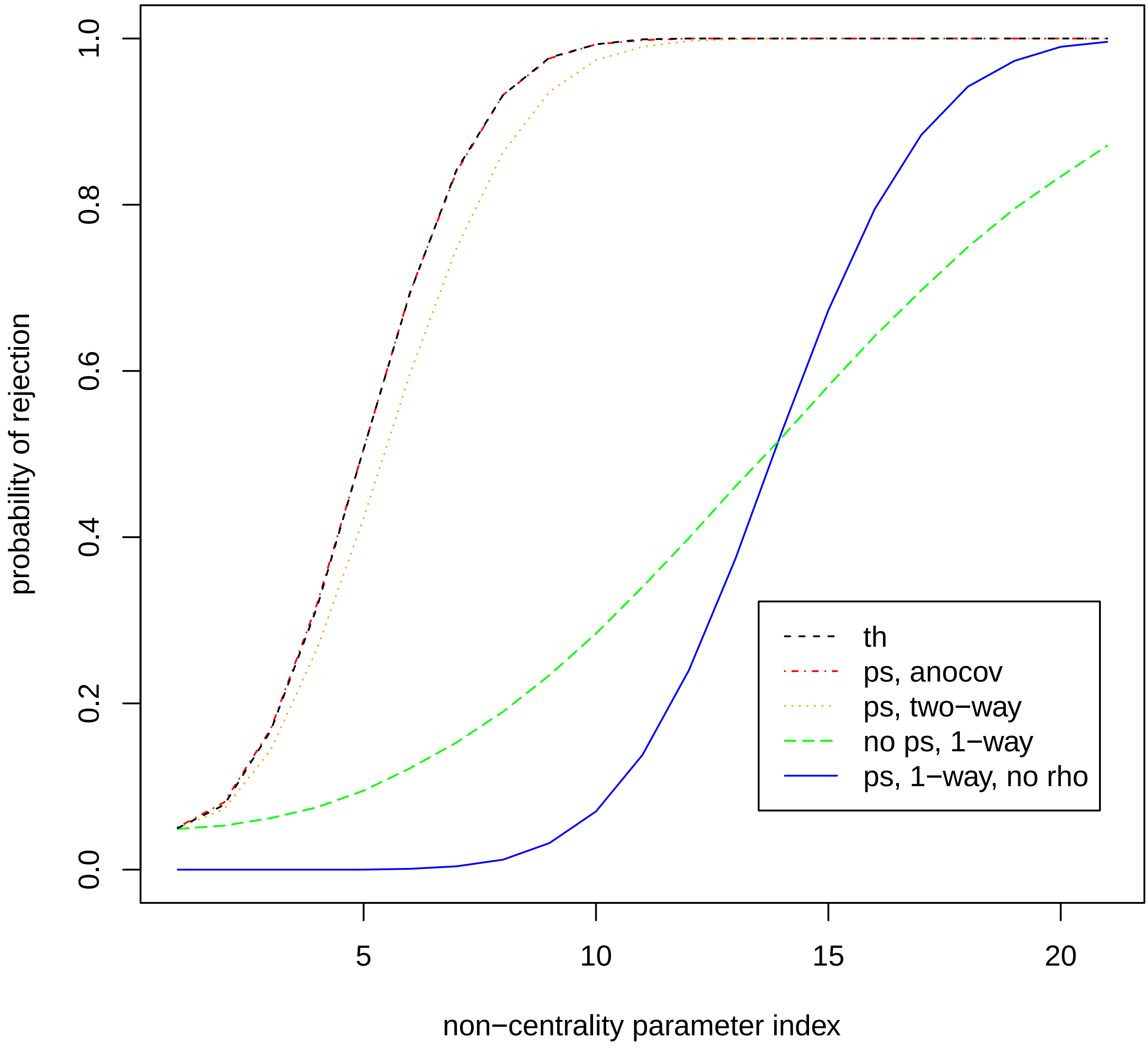

Figure 14: Power plot, $\rho = 0.95$, $J = 2$, $I = 20$.

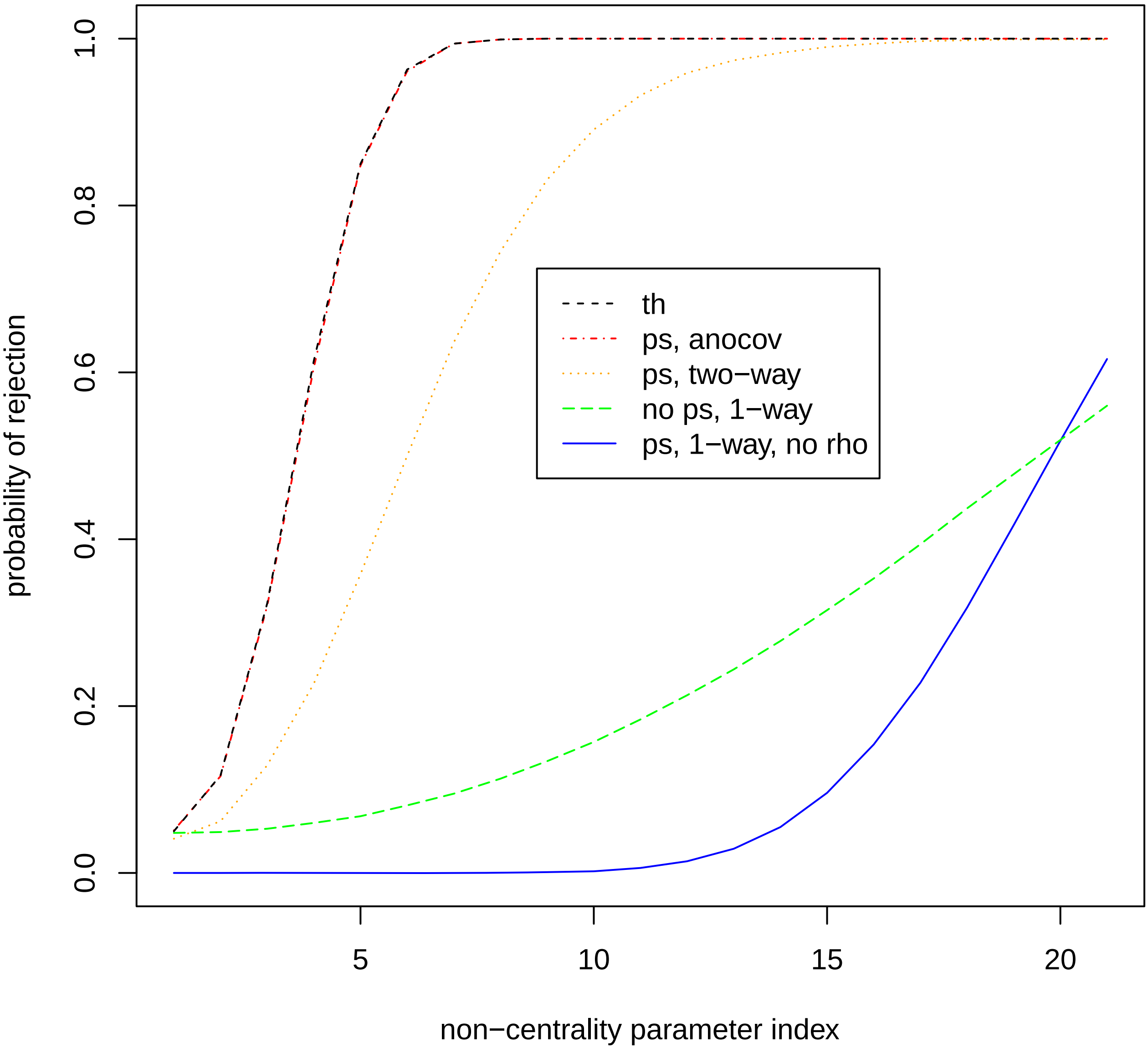

Figure 15: Power plot, $\rho = 0.99$, $J = 2$, $I = 10$.

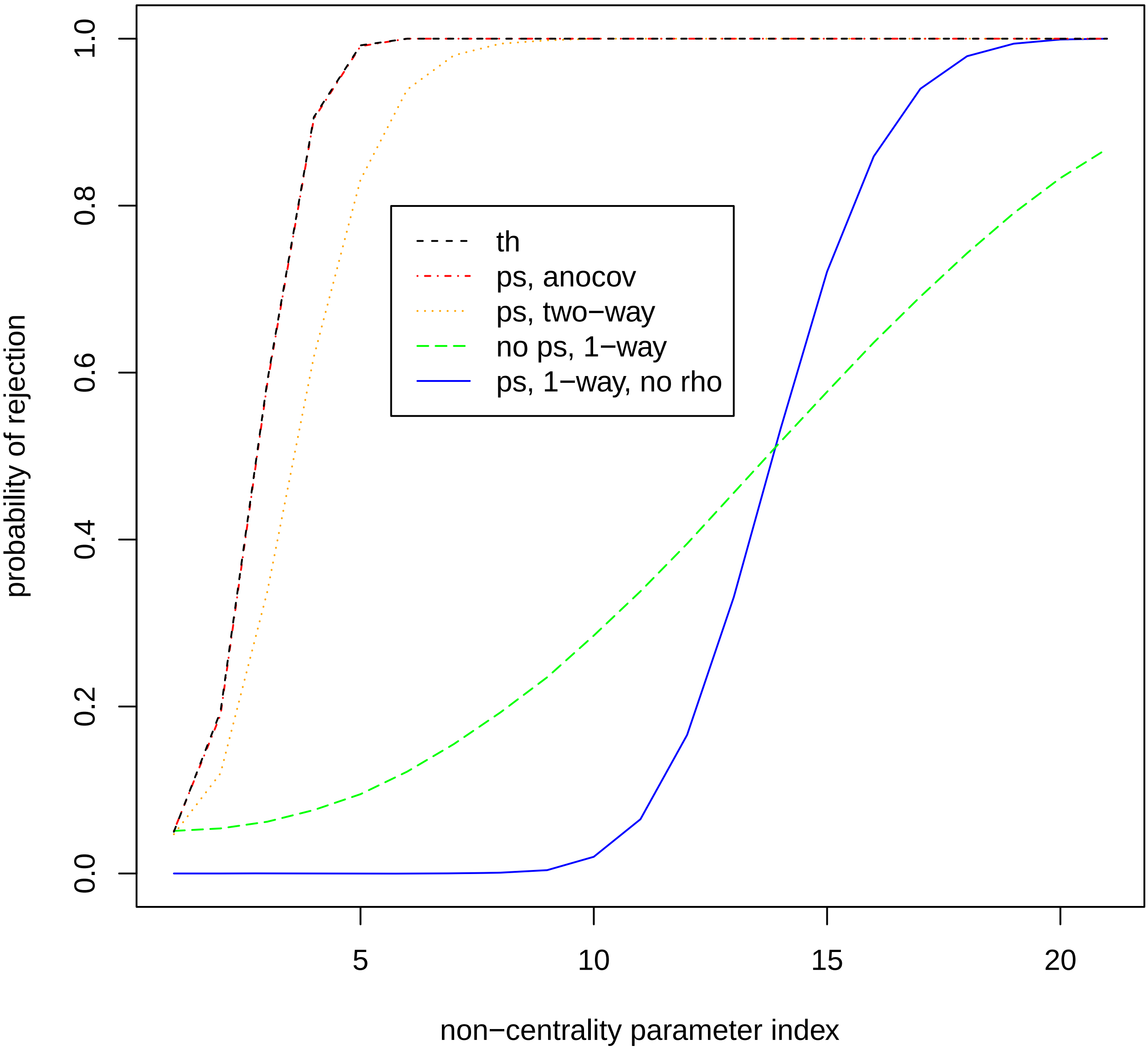

Figure 16: Power plot, $\rho = 0.99$, $J = 2$, $I = 20$.

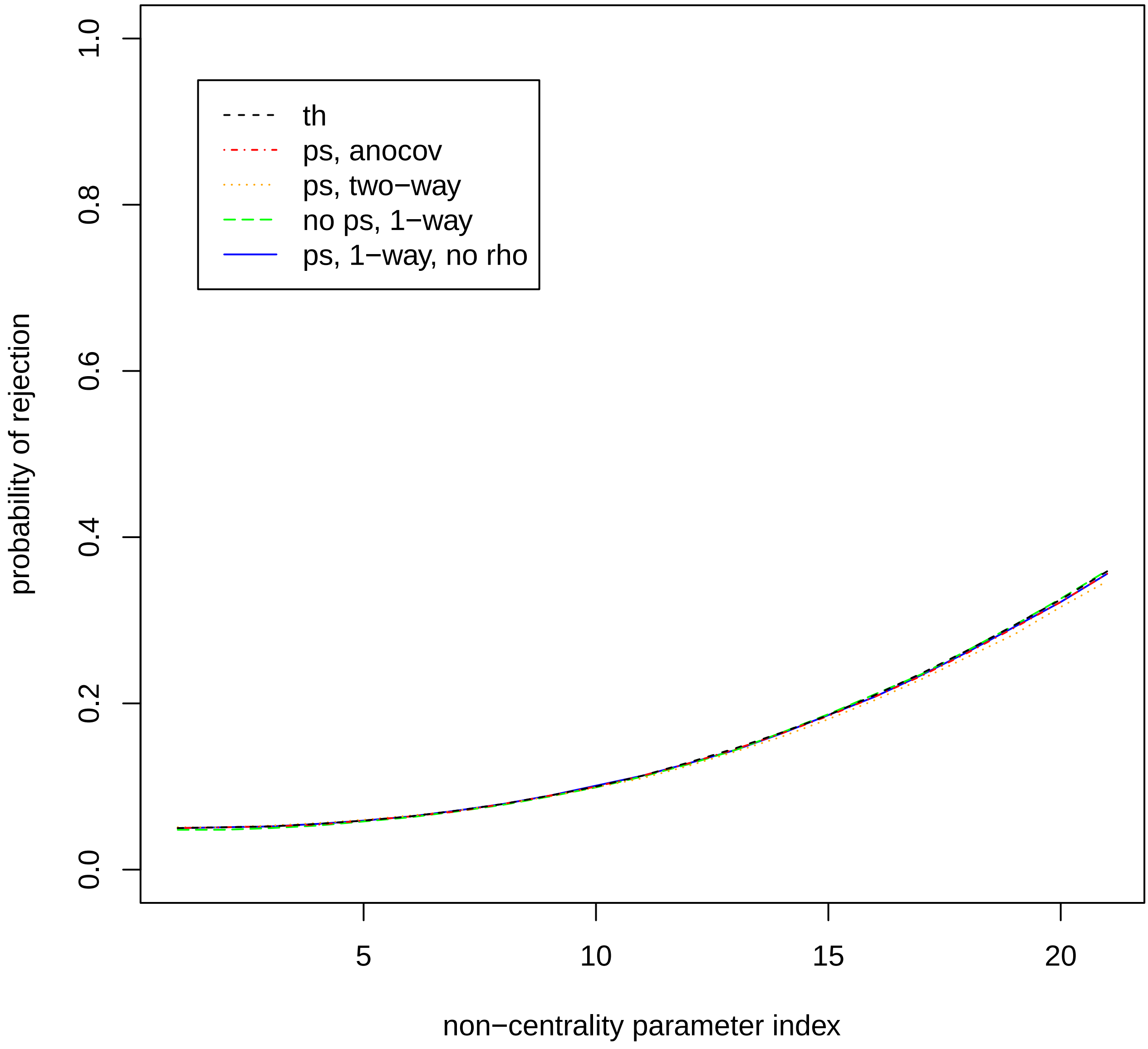

Figure 17: Power plot, $\rho = 0.0$, $J = 5$, $I = 10$.

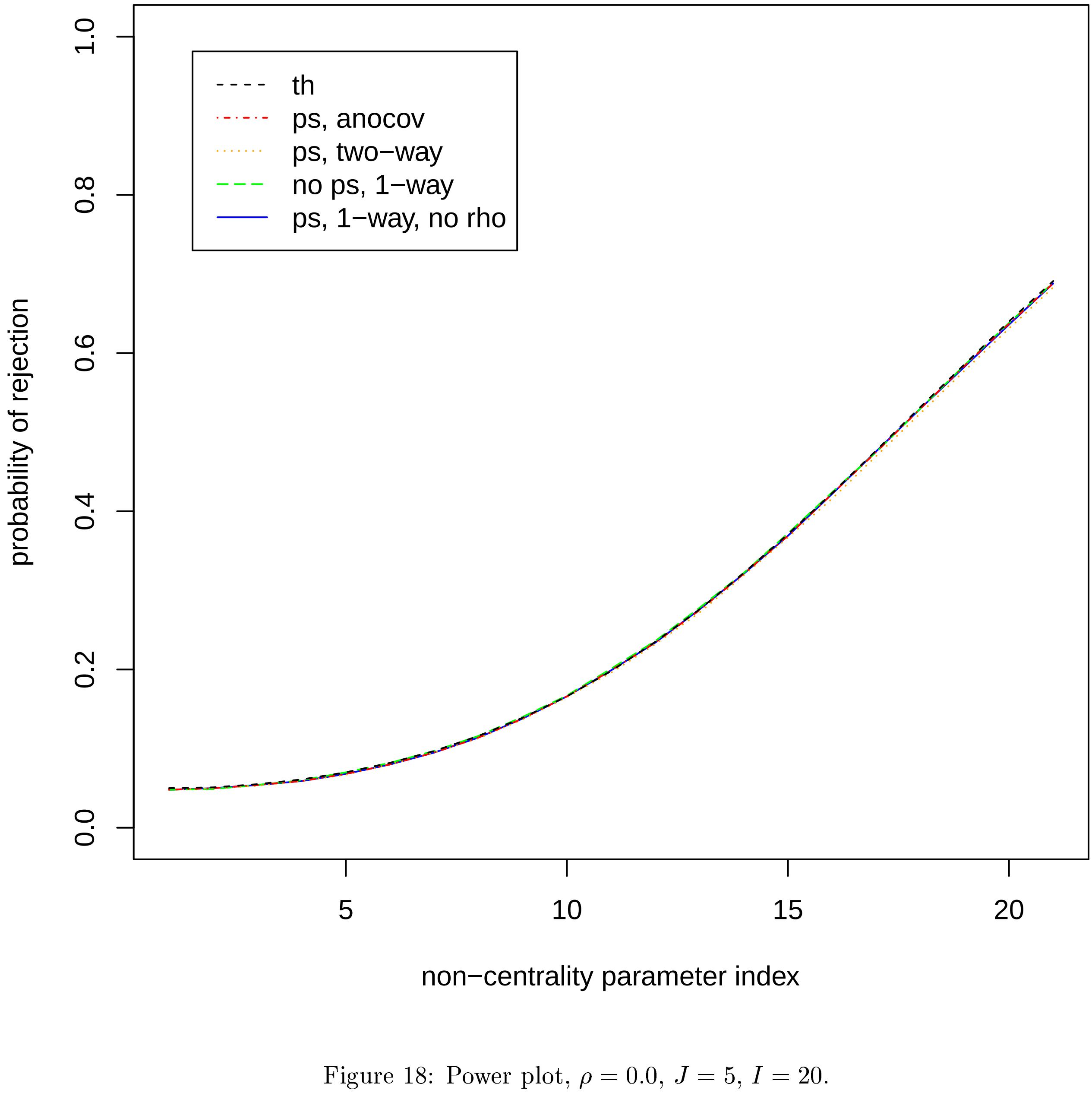

Figure 18: Power plot, $\rho = 0.0$, $J = 5$, $I = 20$.

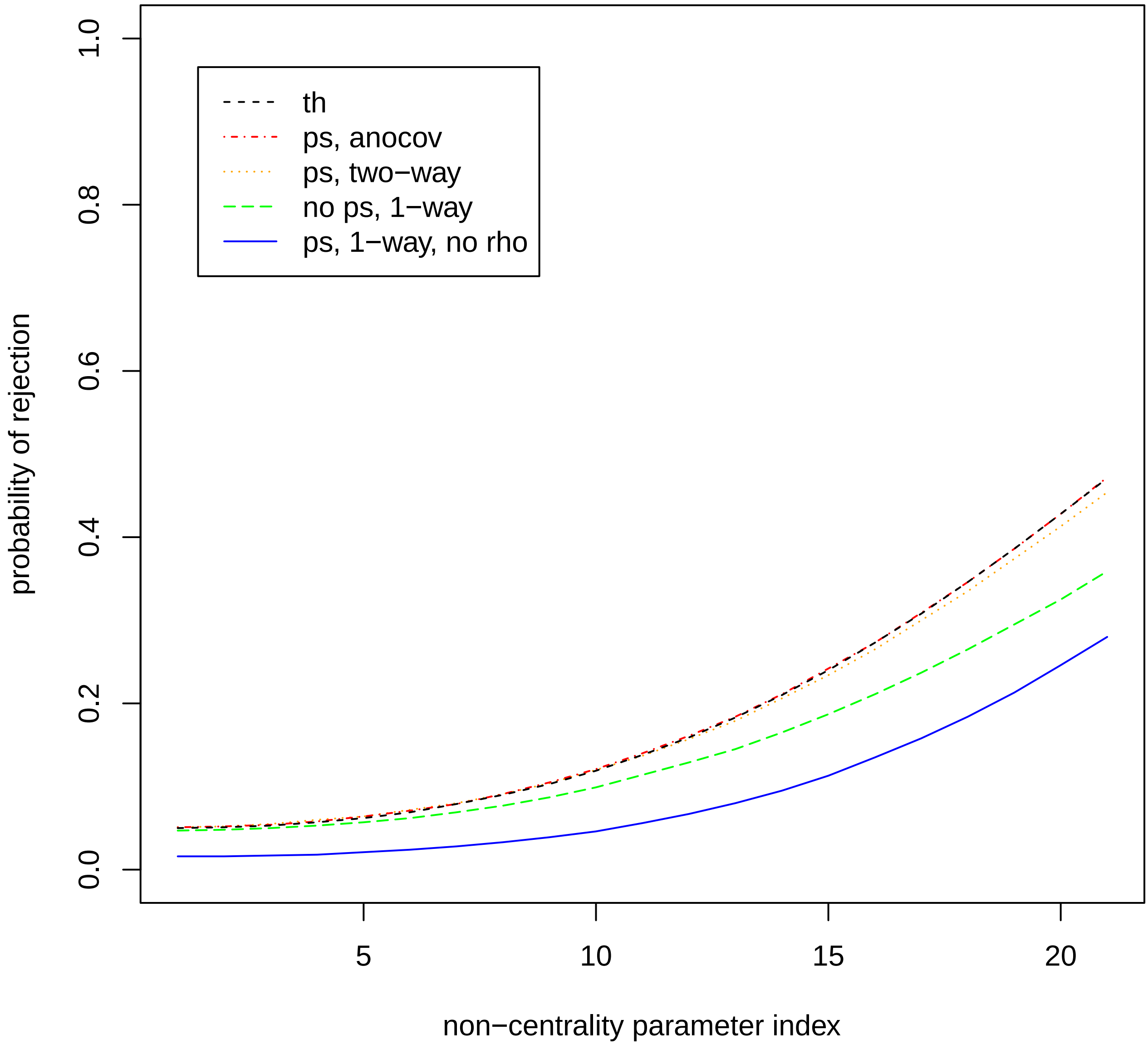

Figure 19: Power plot, $\rho = 0.5$, $J = 5$, $I = 10$.

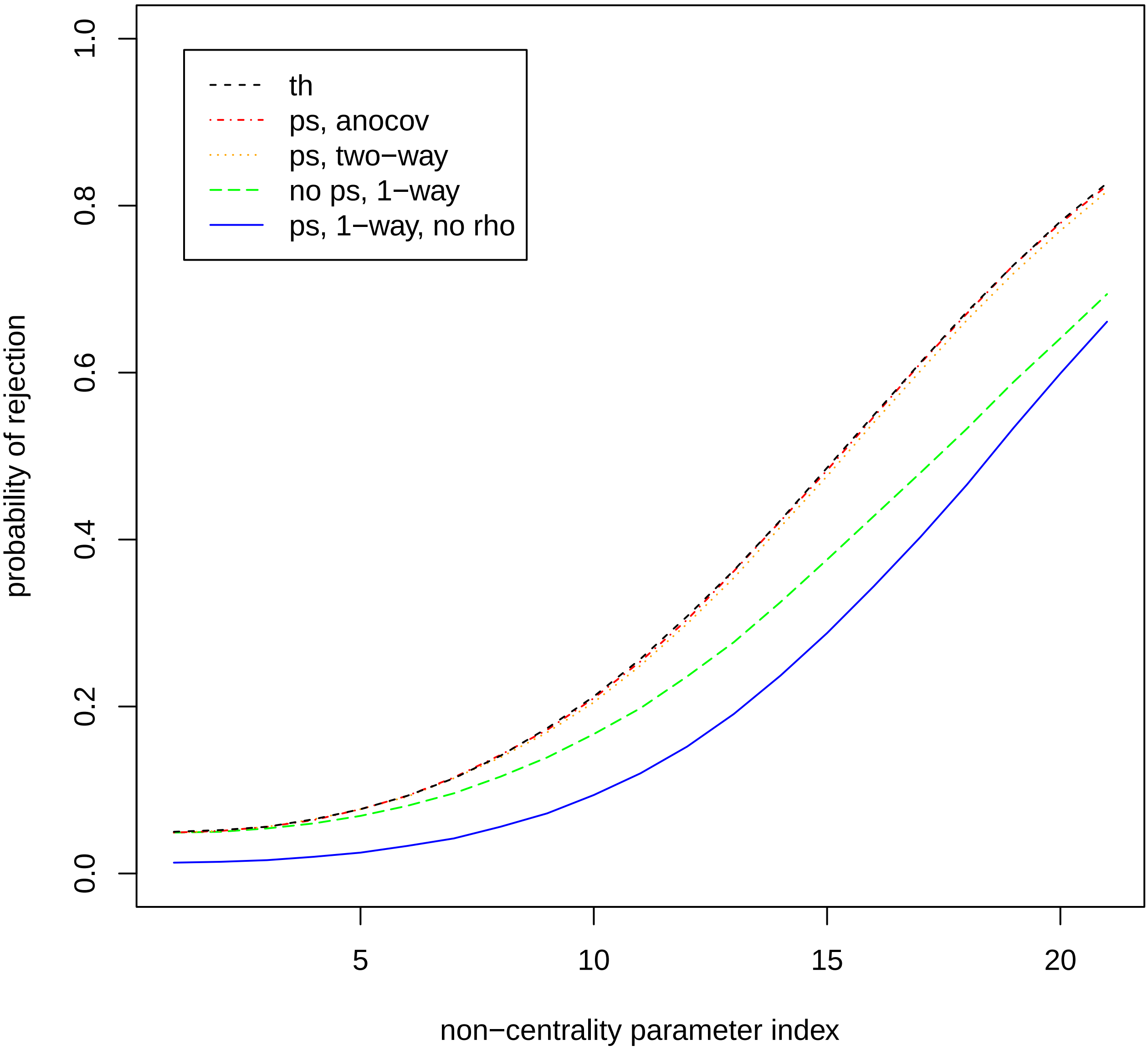

Figure 20: Power plot, $\rho = 0.5$, $J = 5$, $I = 20$.

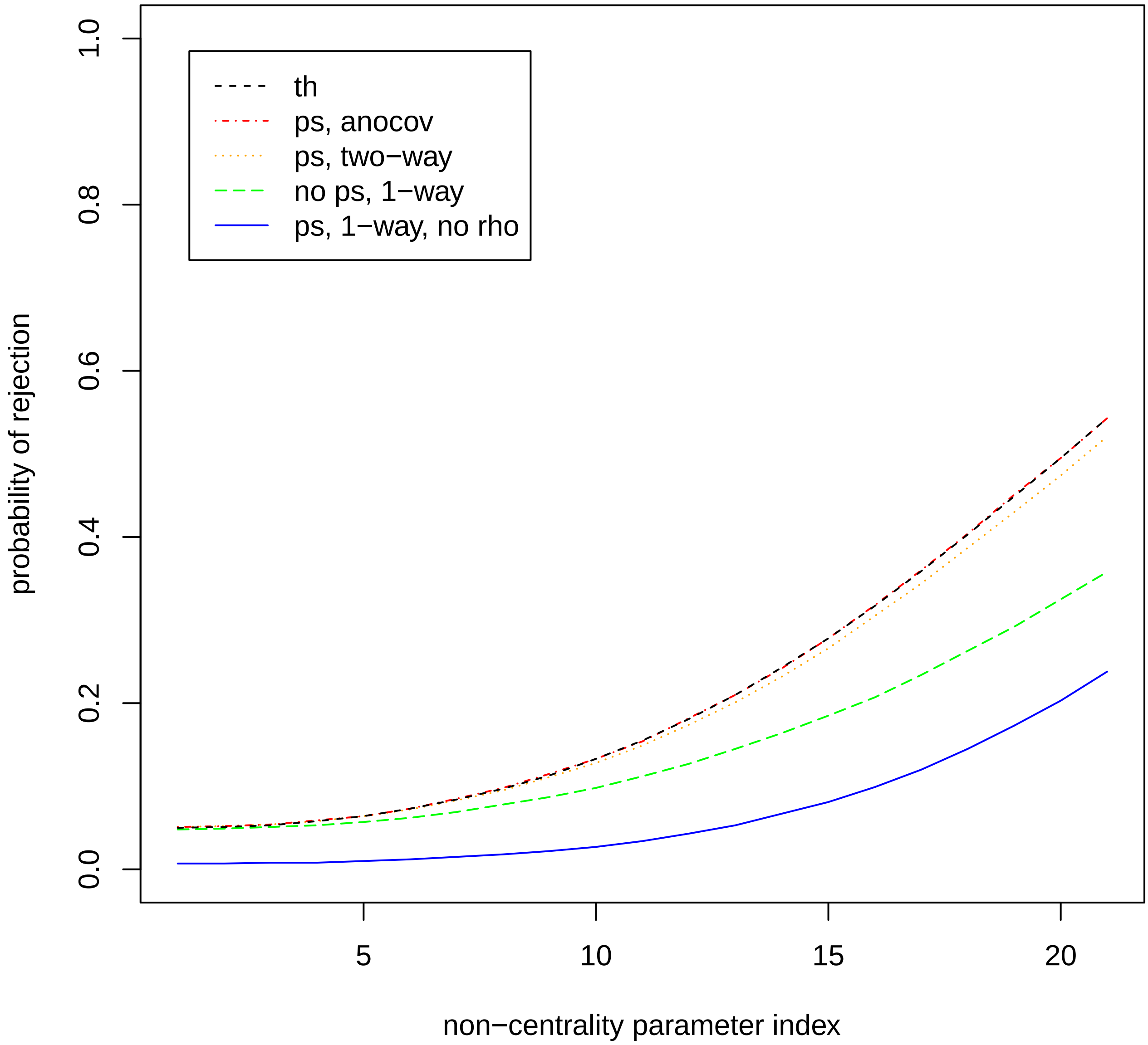

Figure 21: Power plot, $\rho = 0.6$, $J = 5$, $I = 10$.

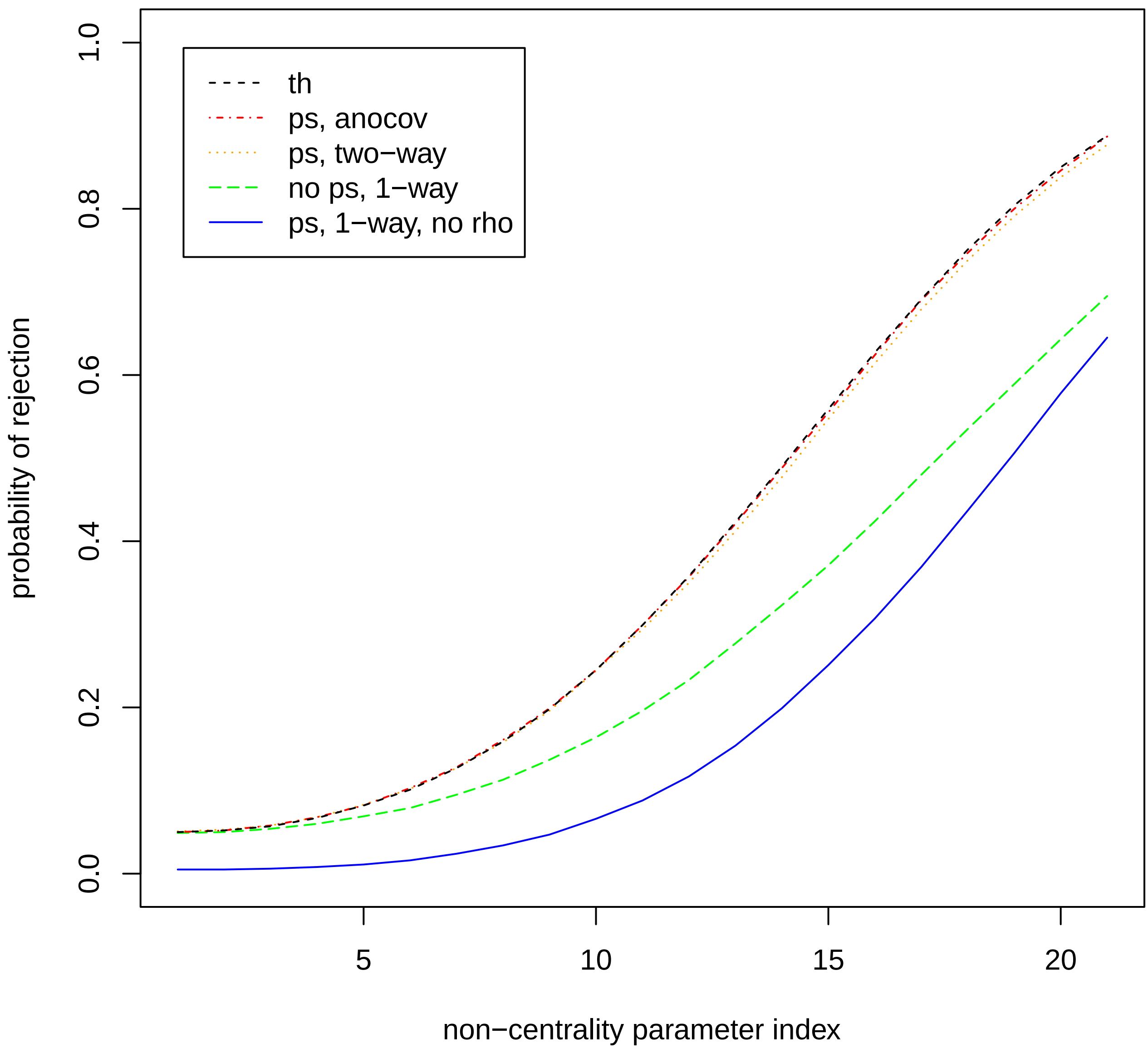

Figure 22: Power plot, $\rho = 0.6$, $J = 5$, $I = 20$.

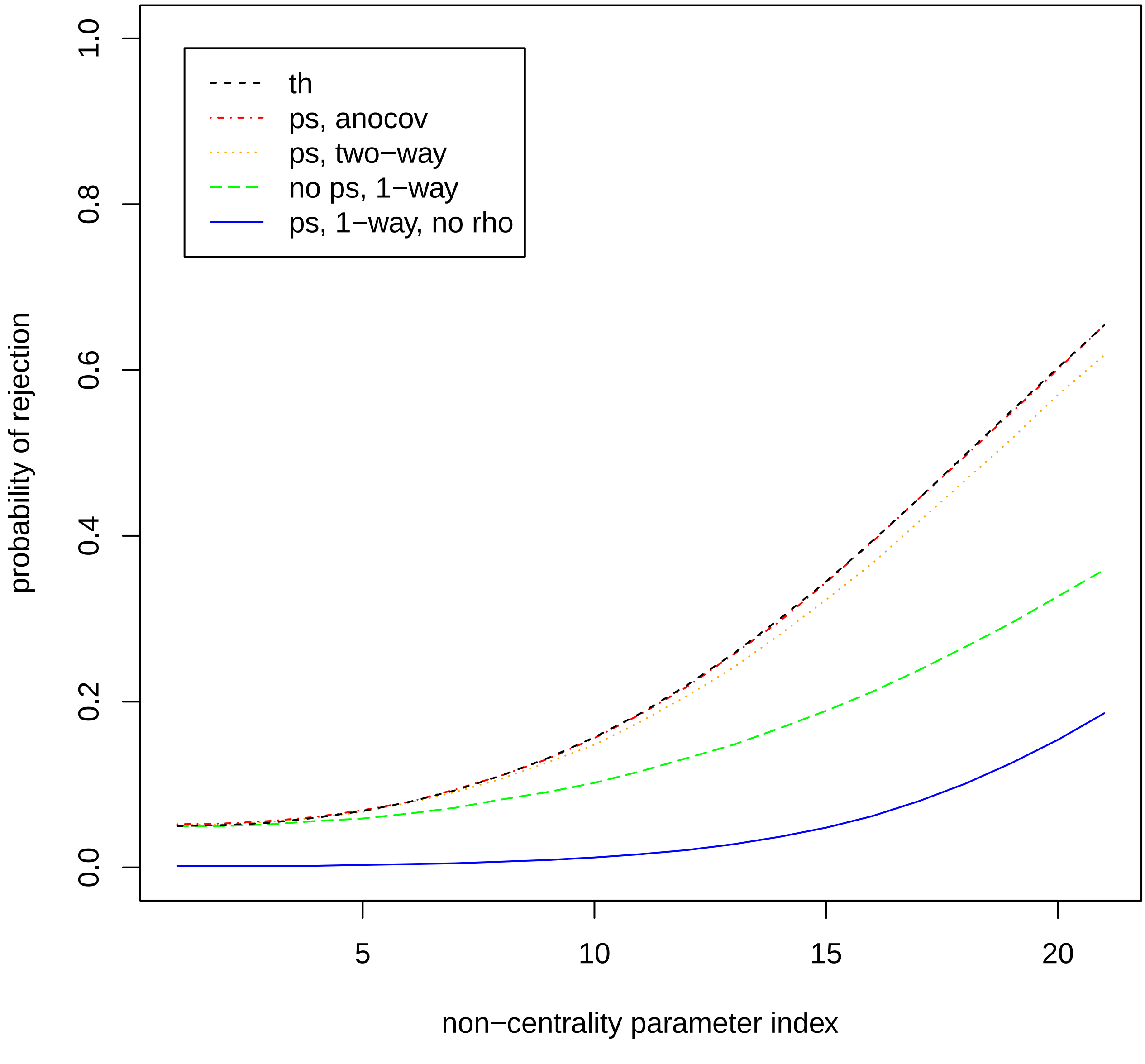

Figure 23: Power plot, $\rho = 0.7$, $J = 5$, $I = 10$.

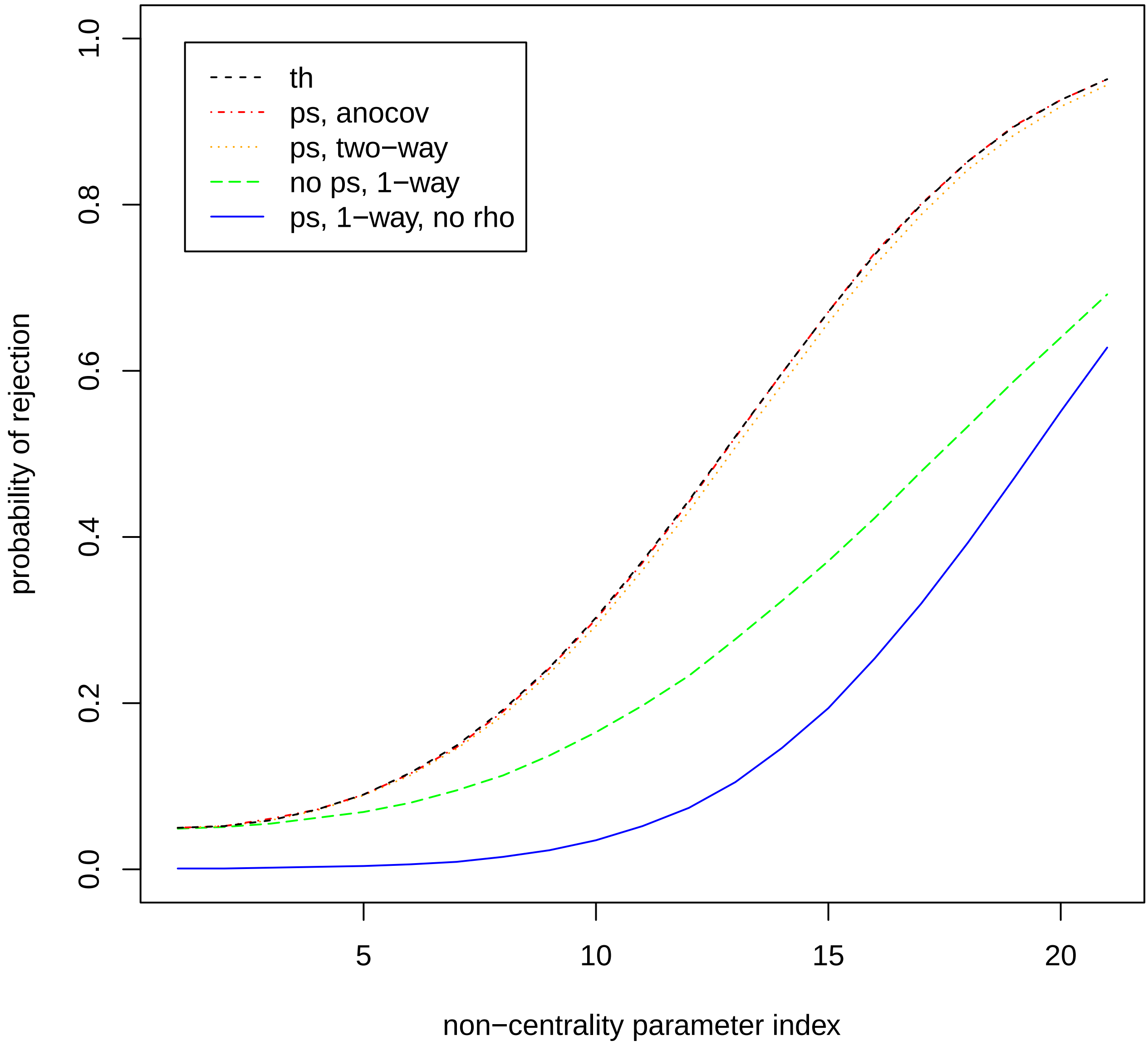

Figure 24: Power plot, $\rho = 0.7$, $J = 5$, $I = 20$.

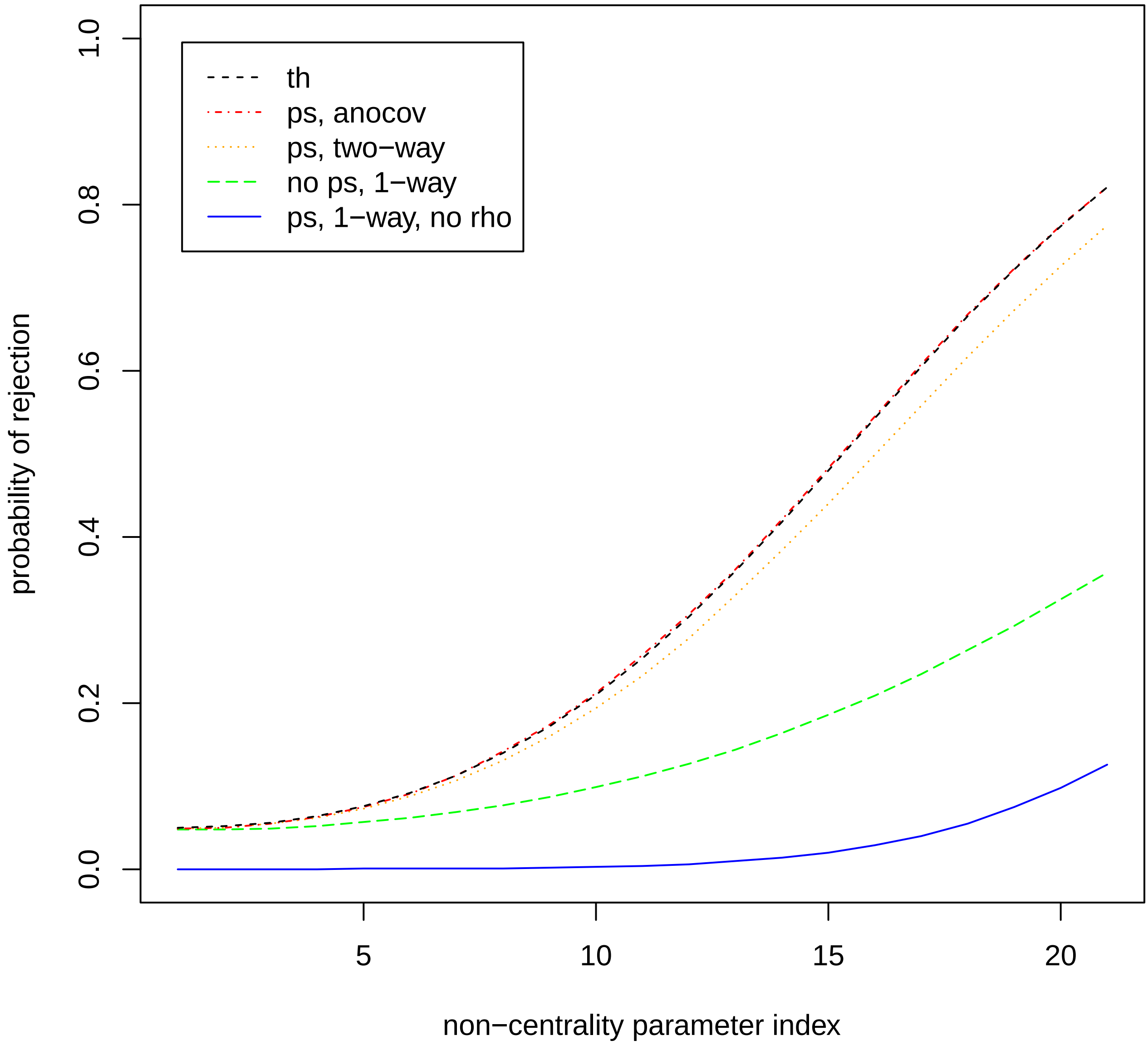

Figure 25: Power plot, $\rho = 0.8$, $J = 5$, $I = 10$.

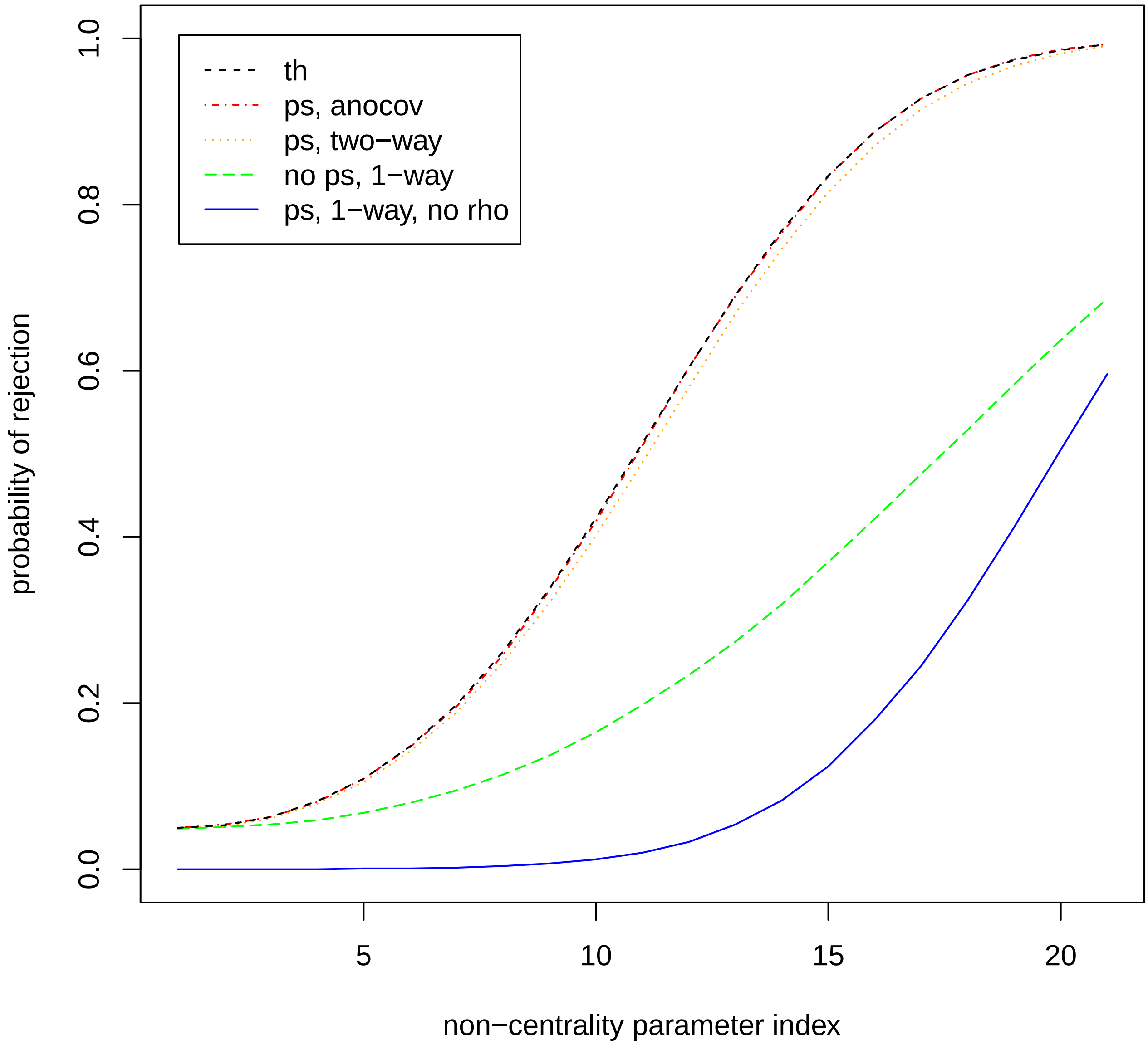

Figure 26: Power plot, $\rho = 0.8$, $J = 5$, $I = 20$.

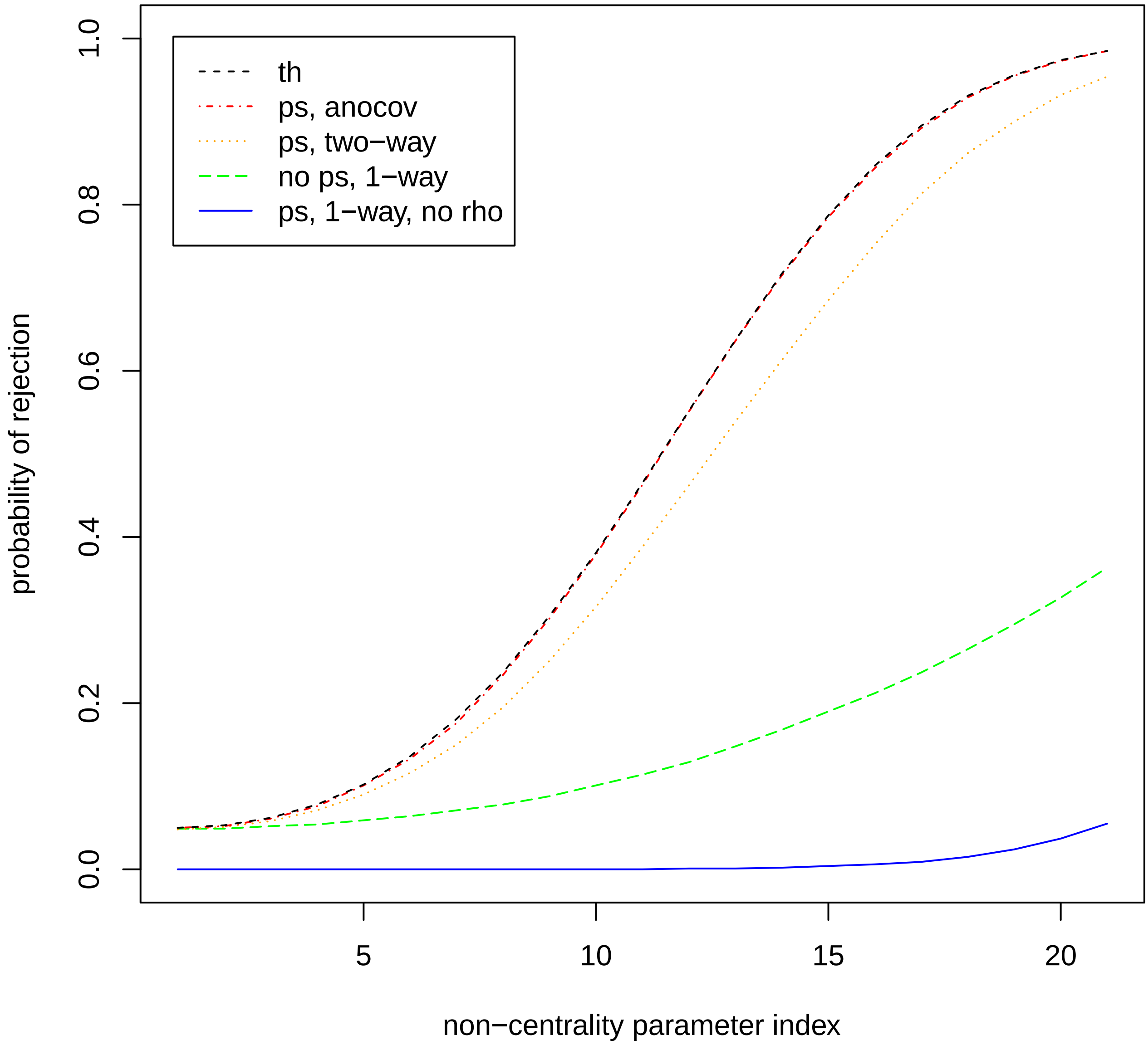

Figure 27: Power plot, $\rho = 0.9$, $J = 5$, $I = 10$.

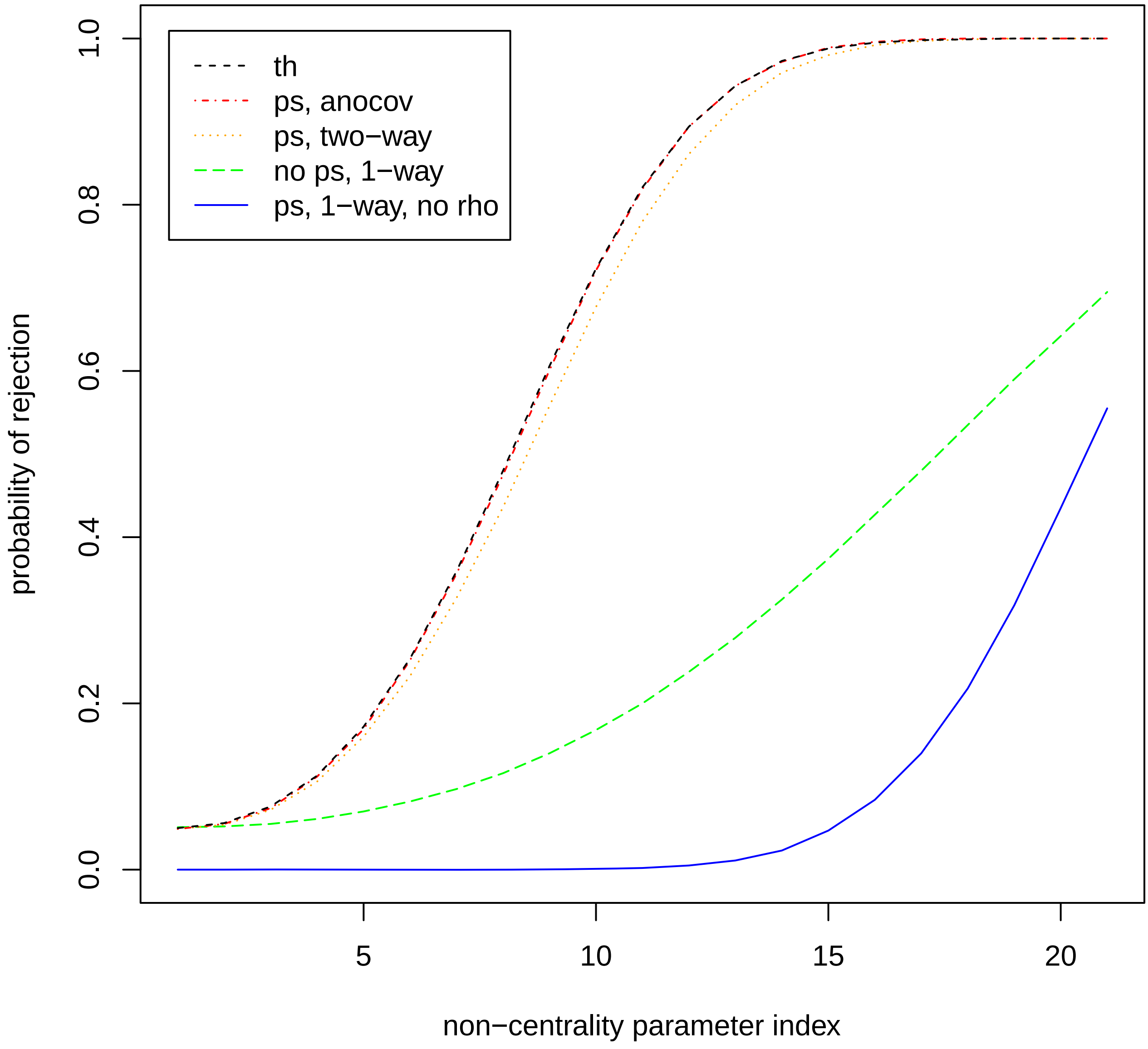

Figure 28: Power plot, $\rho = 0.9$, $J = 5$, $I = 20$.

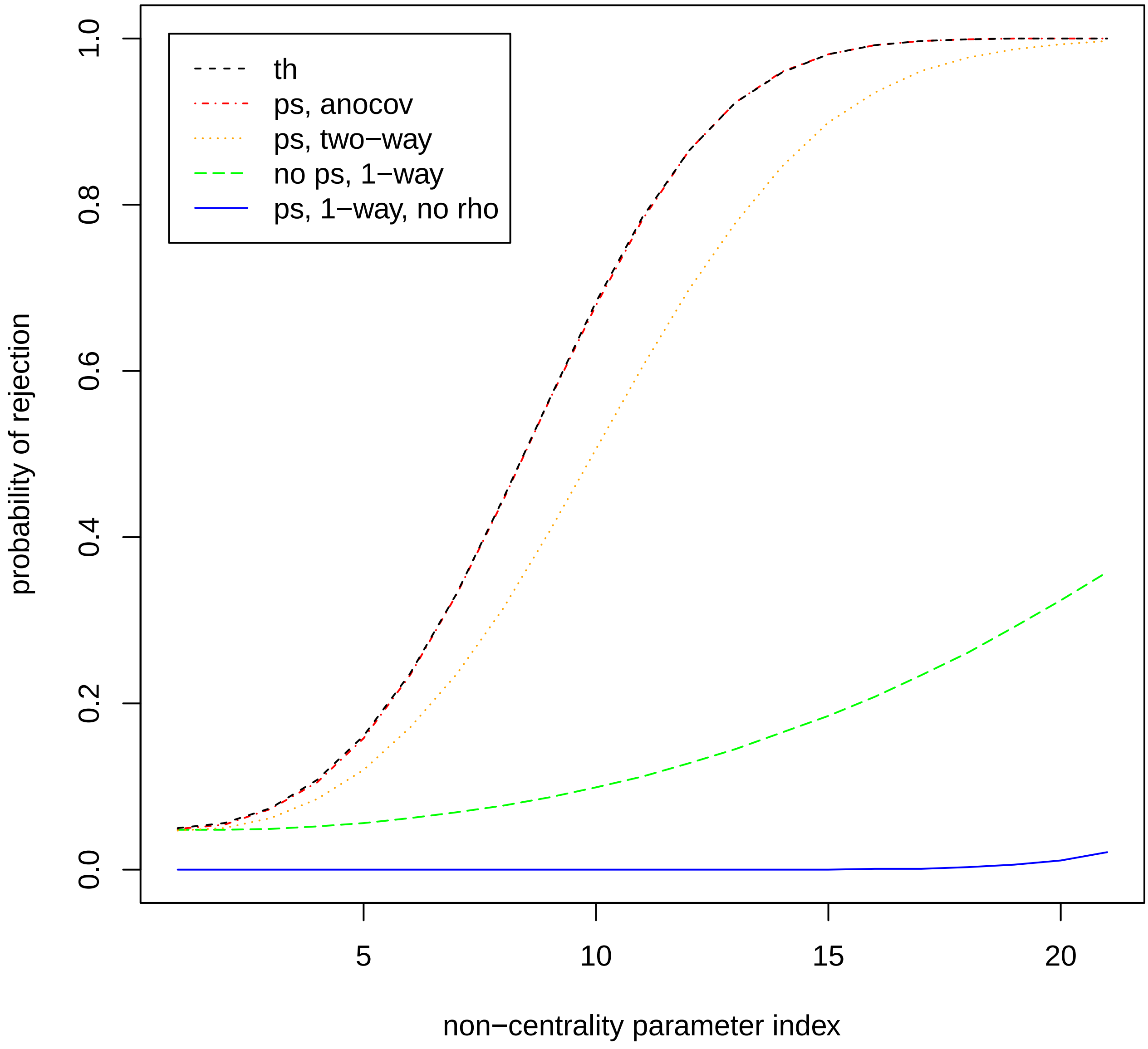

Figure 29: Power plot, $\rho = 0.95$, $J = 5$, $I = 10$.

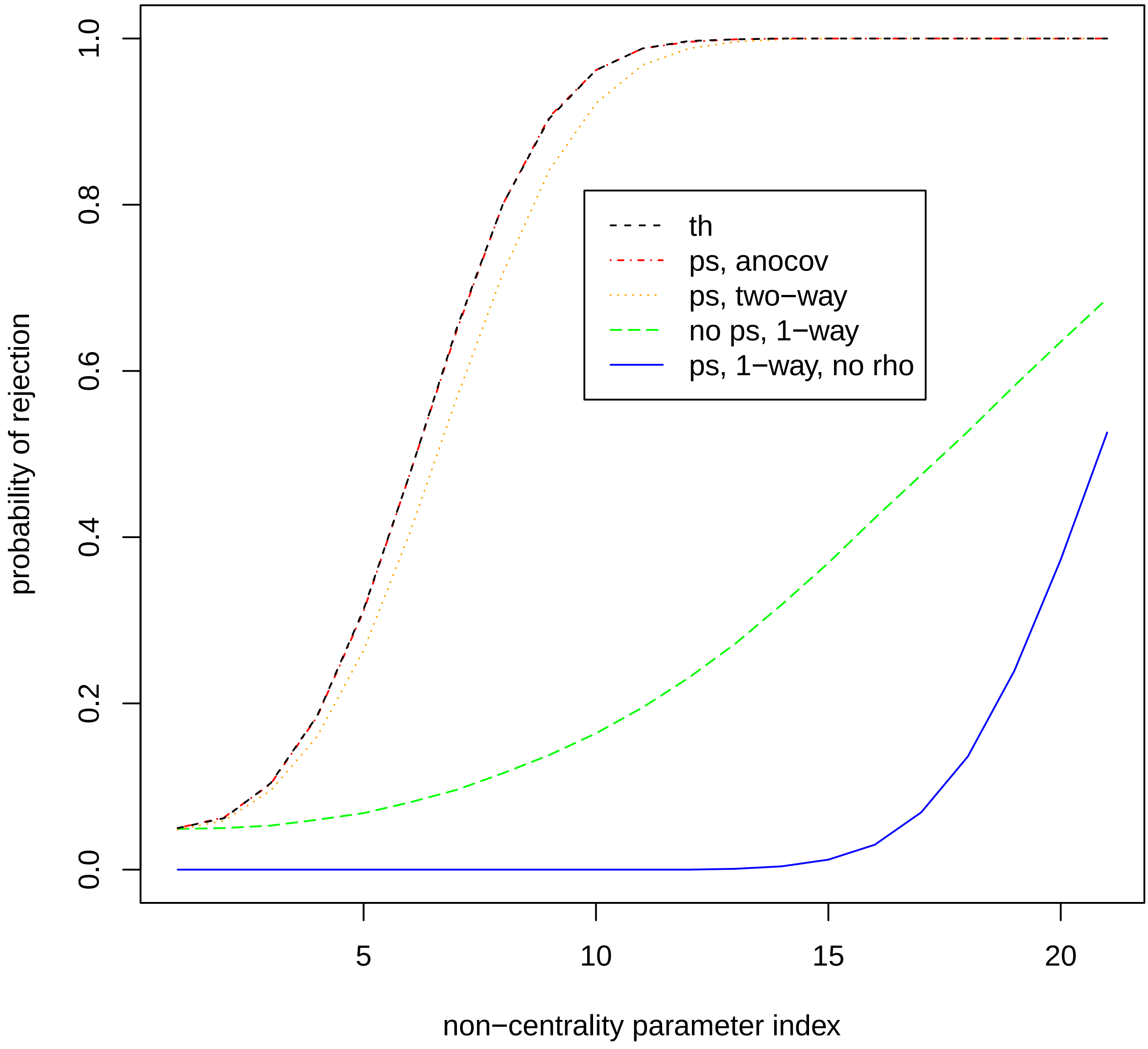

Figure 30: Power plot, $\rho = 0.95$, $J = 5$, $I = 20$.

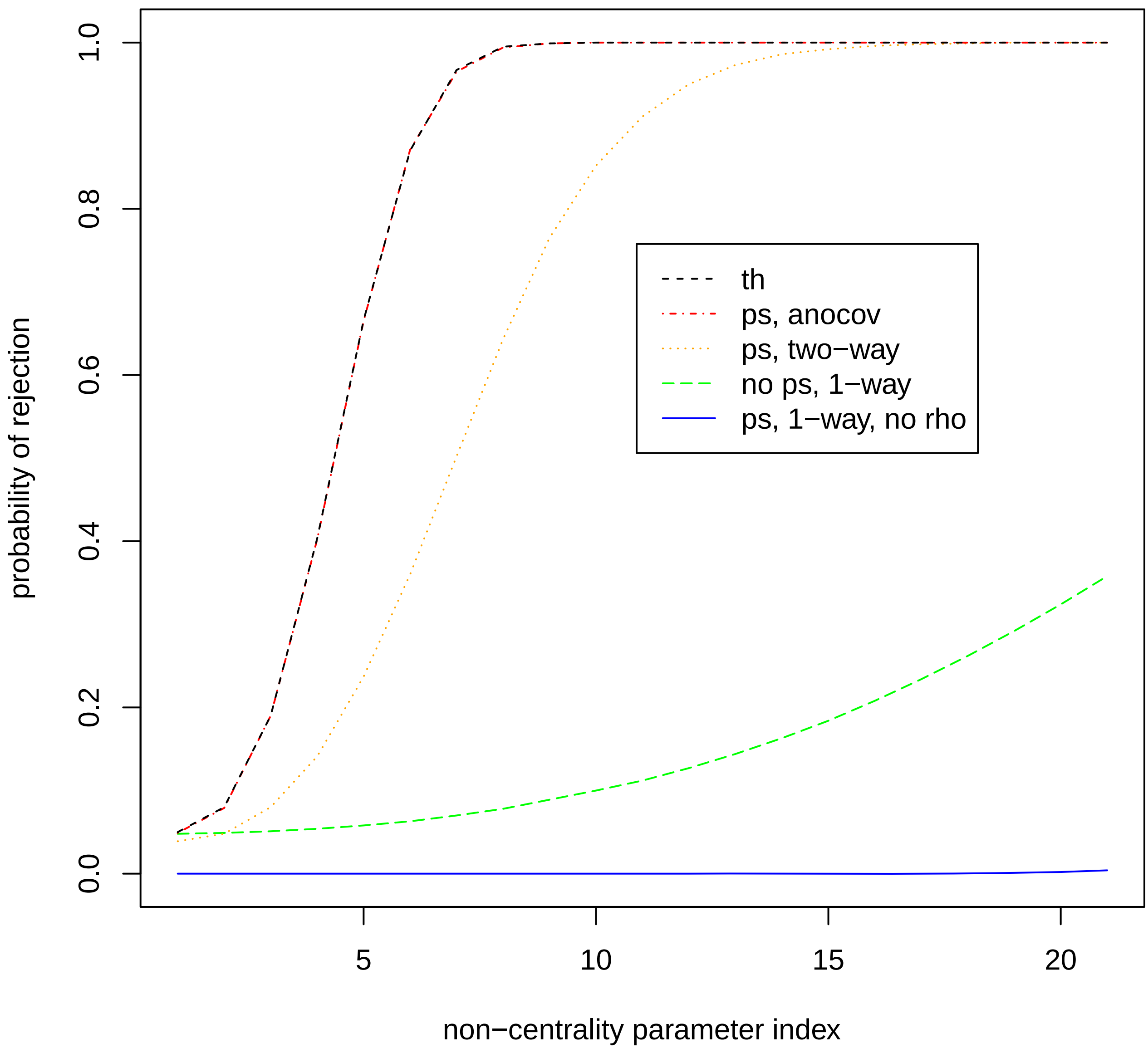

Figure 31: Power plot, $\rho = 0.99$, $J = 5$, $I = 10$.

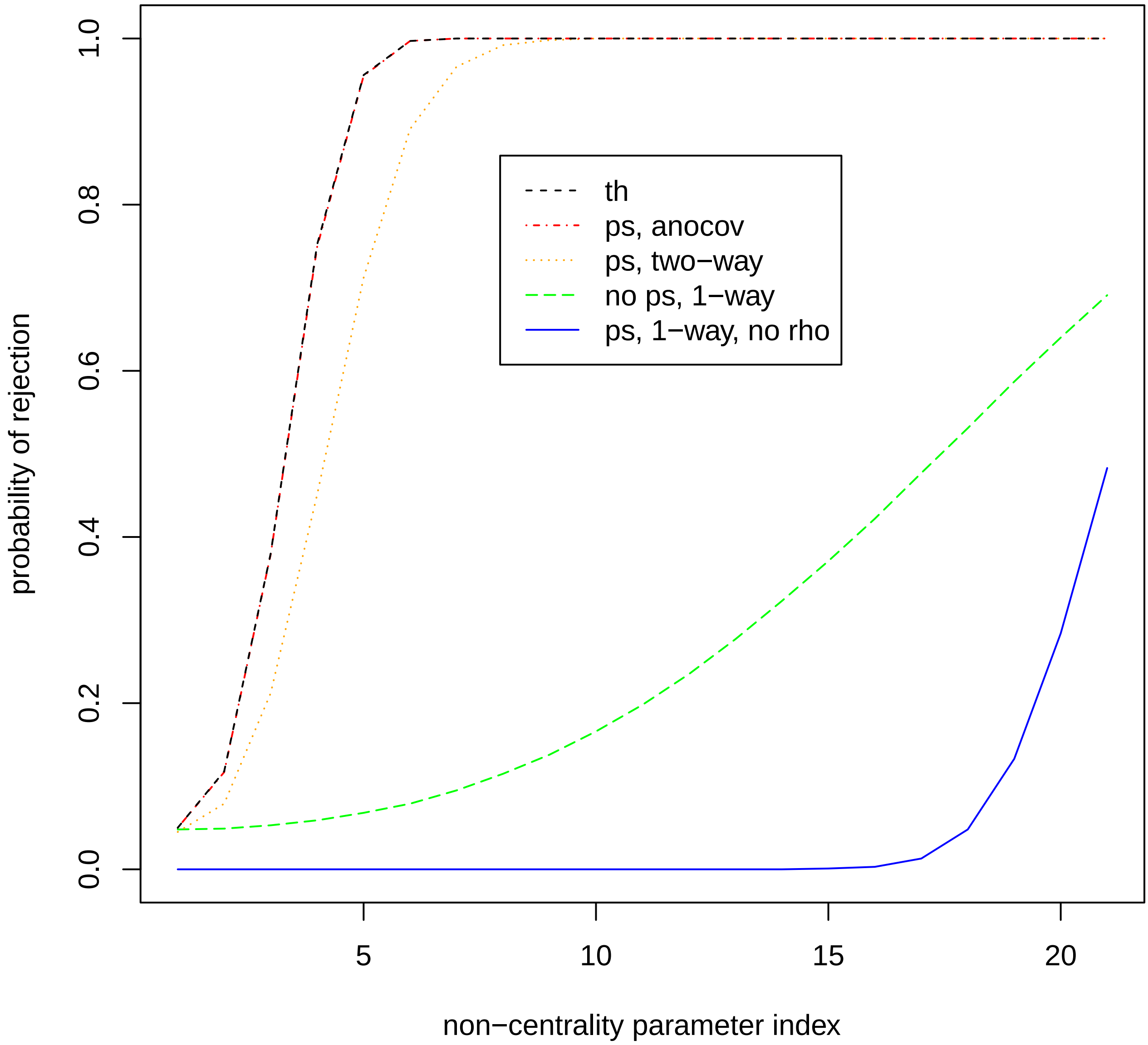

Figure 32: Power plot, $\rho = 0.99$, $J = 5$, $I = 20$.

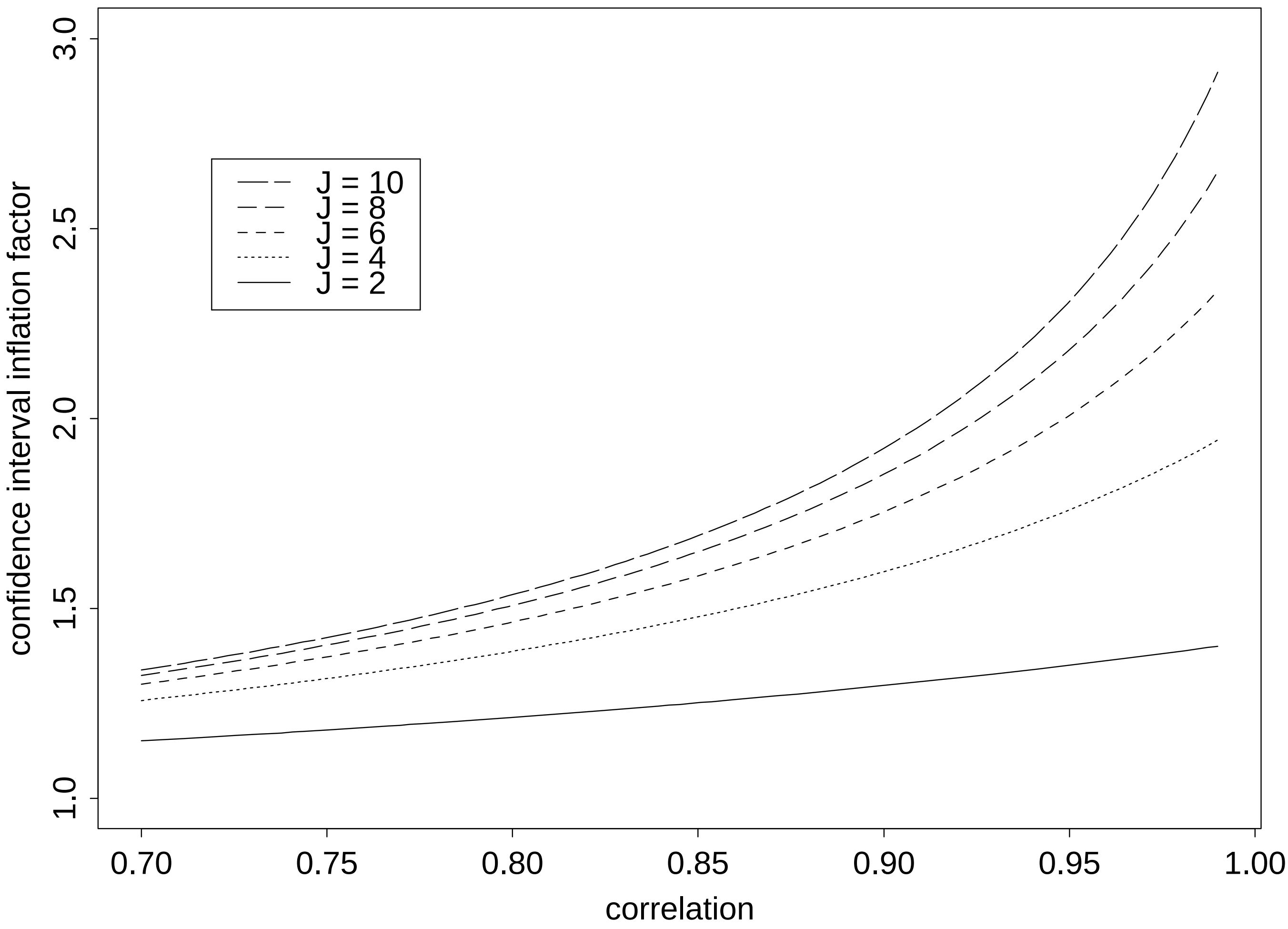

Figure 33: Unblocked ANOVA, Confidence Interval Inflation Factor.

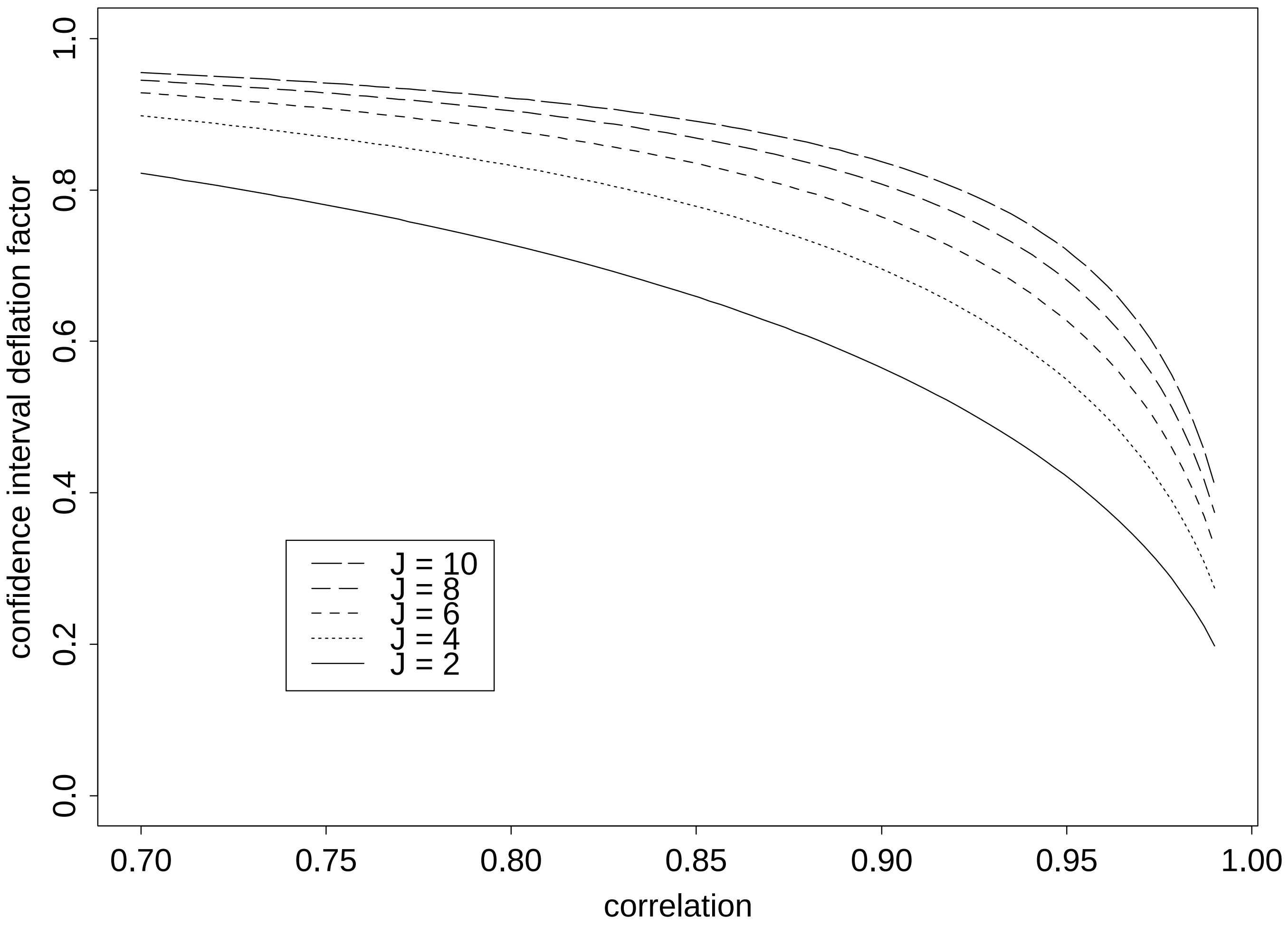

Figure 34: Blocked ANOVA, Confidence Interval Deflation Factor.

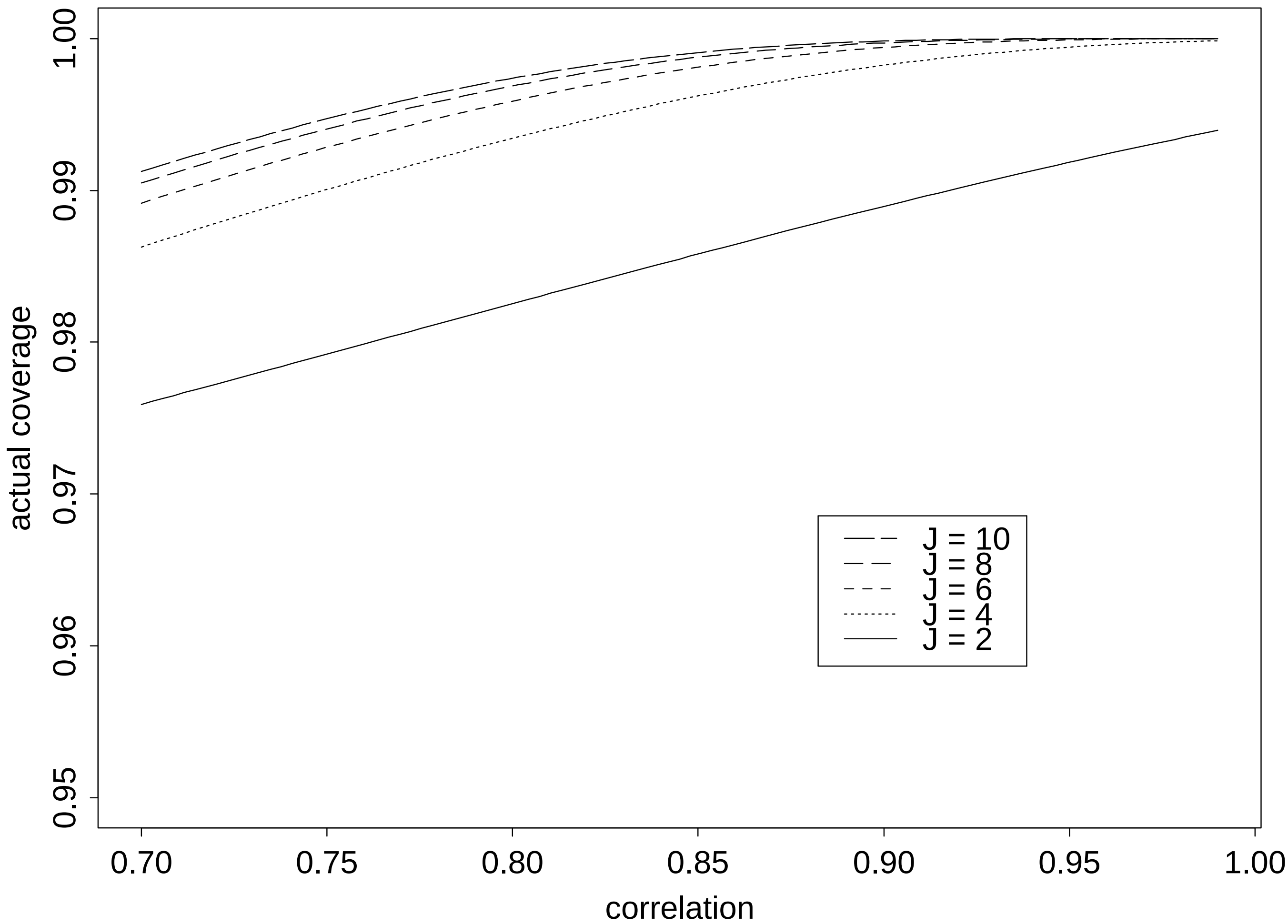

Figure 35: Unblocked ANOVA, Actual Coverage of a Nominal 95% Confidence Interval.

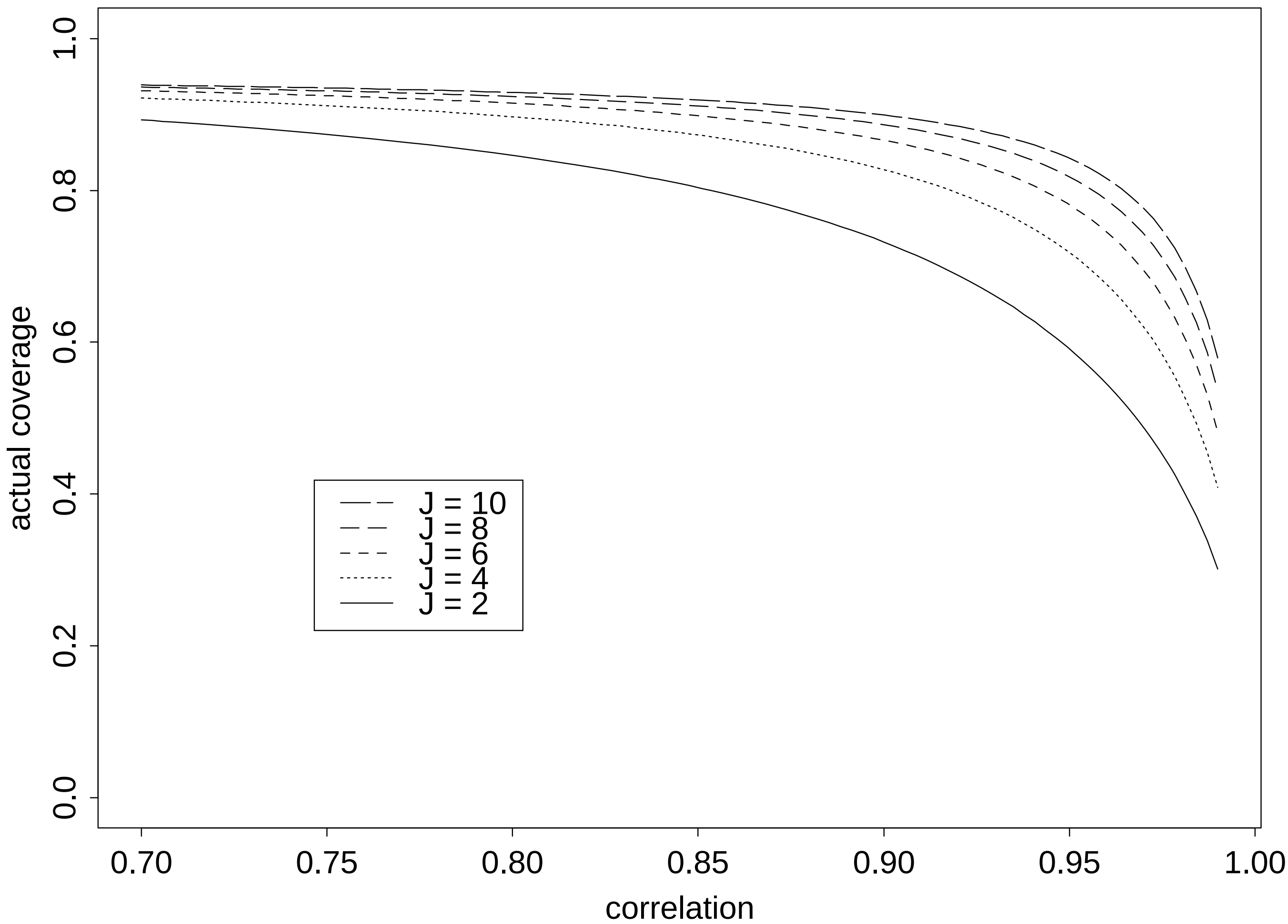

Figure 36: Blocked ANOVA, Actual Coverage of a Nominal 95% Confidence Interval.

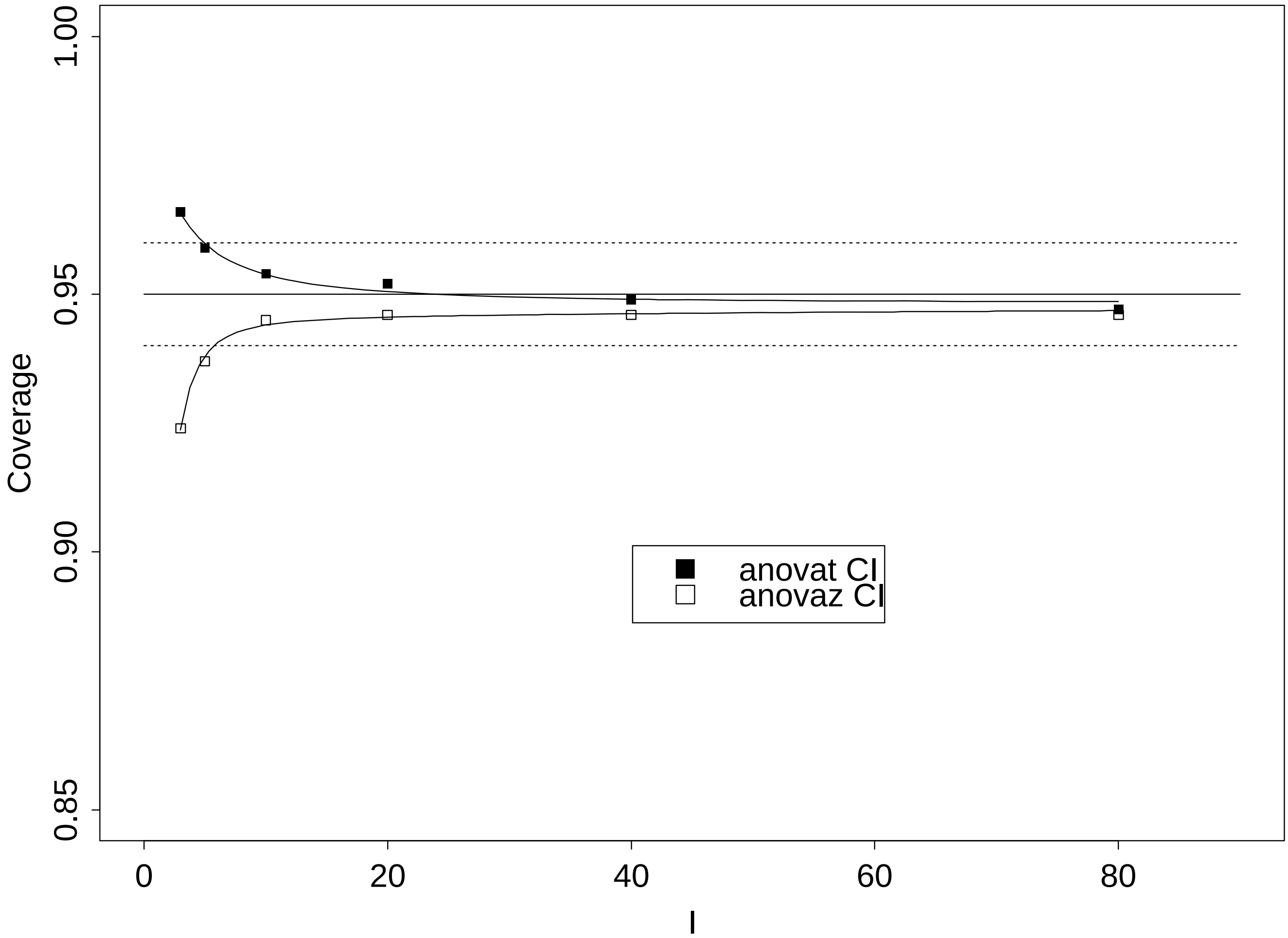

Figure 37: Simulation coverages of anovaz and anovat nominal 95% confidence intervals, $\rho = 0.80, J = 3$. Dotted lines at 0.94 and 0.96.

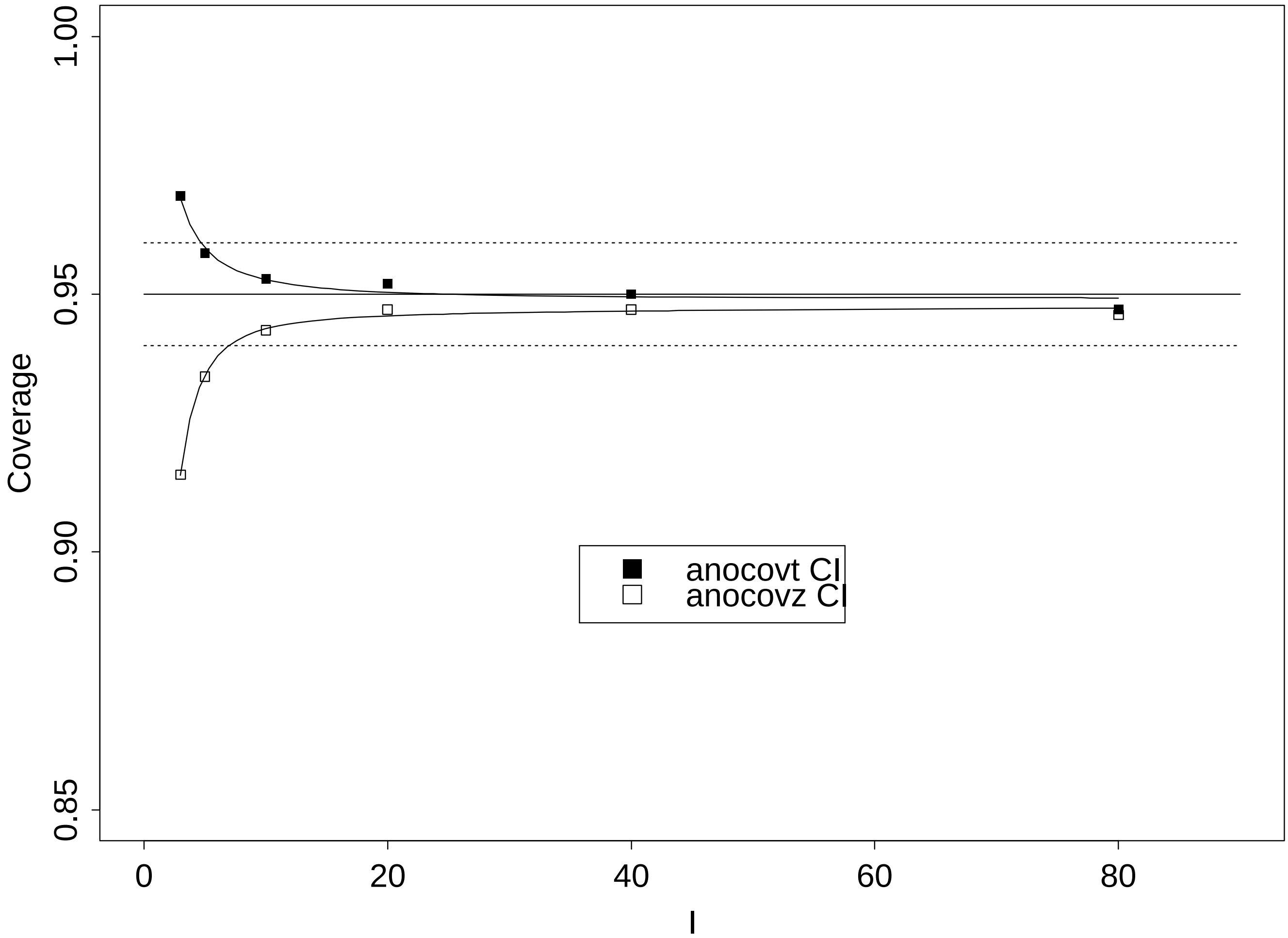

Figure 38: Simulation coverages of anocovz and anocovt nominal 95% confidence intervals, $\rho = 0.80, J = 3$. Dotted lines at 0.94 and 0.96.

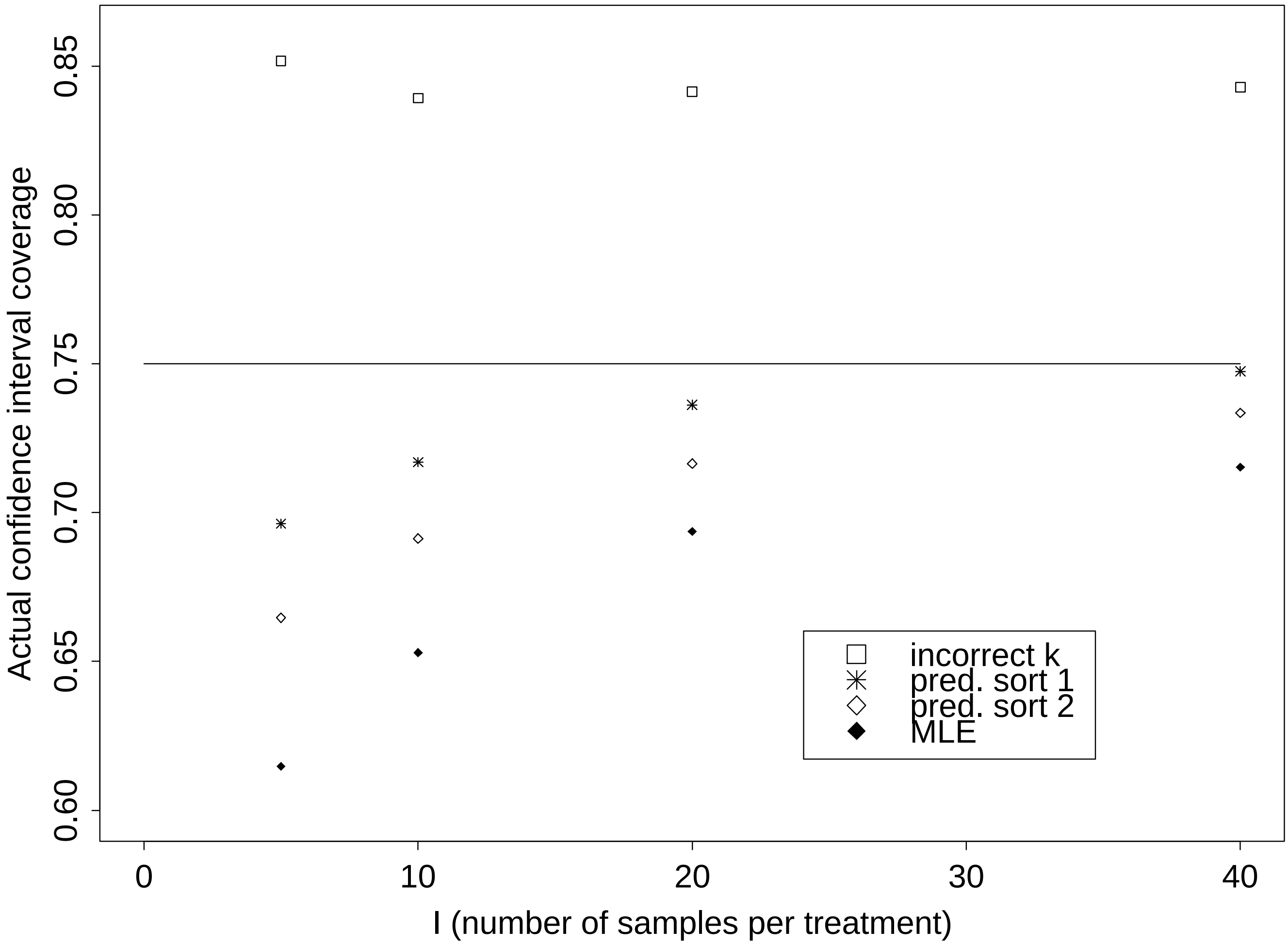

Figure 39: Approach of actual confidence interval coverage to nominal (.75) coverage as sample size increases. For this figure, the correlation between the predictor and the response was 0.85. The number of treatments, J, was four. The confidence interval was for the 0.01 quantile. (See Table 2 in Verrill *et al.* 2004.)

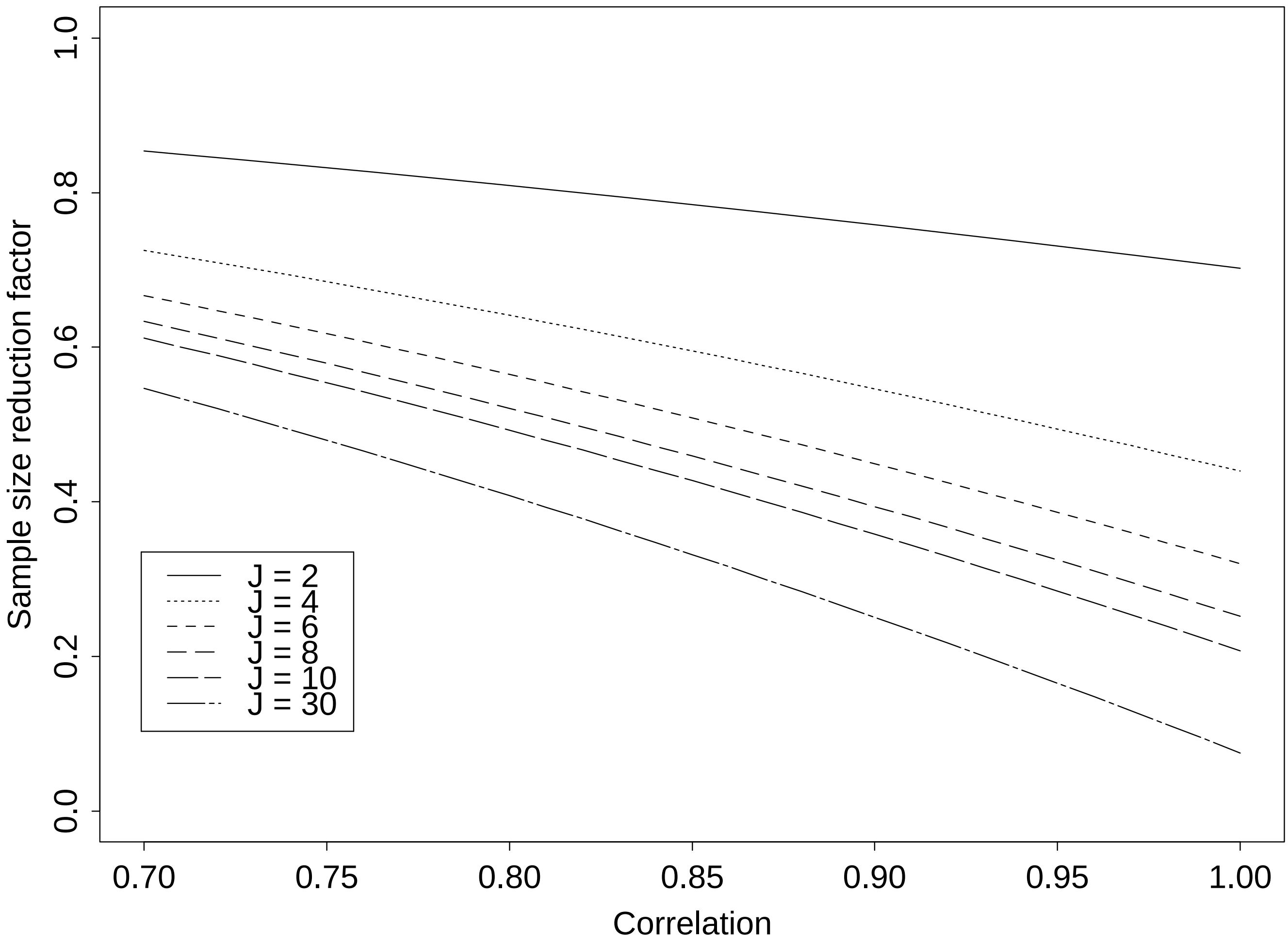

Figure 40: For a confidence bound on the 0.05 quantile, sample size reduction factor as a function of the correlation between the predictor and the response, and the number of treatments J.